Introduction to the Standard Model of Particle Physics – Part 1

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July 31, 2019

Abstract

Despite its limits, the so-called Standard Model describes large parts of Particle Physics in very exact accordance with experiments and is very well tested. Alex Flournoy from the Colorado School of Mines held 29 lectures in 2018 covering this topic. His lectures were available on YouTube at the time this transcript has been assembled and may still be available today as "Particle Physics (2018)" in 29 separate videos. The first 16 lectures cover the theoretical side of the topic with the formal structure of the Standard Model, symmetries, gauge fields and the Higgs mechanism, and the remaining 13 lectures touch on the practical side with the computational aspects of decays and scattering amplitudes. A thorough basis of modern physics including Special Relativity and Quantum Mechanics is required, but knowledge of Quantum Field Theory is not a prerequisite.

1 Introduction

1.1 The Framework for Particle Physics

The topic covered here is the Standard Model of Particle Physics, and that is physics at its deepest level where physicists are completely confident. String theory, for example, is a level deeper, but there is less confidence, and there is very much confidence in chemistry, but it is not as deep. The Standard Model of Particle Physics is incredibly well tested and is extremely accurate except for some puzzles still to be addressed.

The following high-level language is applicable to everything in physics and categorizes what is done in various areas of physics:

- A *framework* is used for describing the evolution of a system.
- In a *theory* a chosen framework is applied to a context.
- A *model* is an effective theory that requires some inputs that are not predicted by theory itself.

Classical Lagrangian dynamics is an example of a framework and string theory is a theory, but it turns out that one is mostly concerned with models in physics. The Standard Model, as the name suggests, is a model. It is not a pure theory but an effective theory with inputs coming from many experimental measurements such as the four dimensions of spacetime and the measured electron mass. String theory is different. It has no experimental inputs but deduces everything from mathematical consistency including the dimensionality of spacetime. Working with frameworks alone is not very useful because one wants to study something such as a real physical system, but frameworks are needed as a basis. Two important frameworks in the history of physics are Newtonian mechanics and the framework of Special Relativity.

The framework of Newtonian physics has the following characteristics:

- It is usable for low speeds.
- It is based on spacetime modeled as a three-dimensional space and an absolute time.
- The underlying principle of relativity is the Galilean principle of relativity.

In contrast, the relativistic framework based on Special Relativity has the following characteristics:

- It is usable for any speed.
- It is based on the three- plus one-dimensional spacetime.
- The underlying principle of relativity is the one of Special Relativity.

Special relativity[1] has the two important properties that it is basically the same as the Galilean principle of relativity for speeds which are small compared to the speed of light, and that the speed of light is the same in all frames.

Because speeds in Particle Physics with today's accelerators are in the order of 99.9999% of the speed of light, the Newtonian framework is not appropriate. Even with smaller speeds it would not be wise to base a newly developed theory or model on a limited framework. Thus, Special Relativity is the framework for Particle Physics. The transition from Newtonian mechanics to relativistic mechanics is pretty smooth. However, this is different for the transition from Newtonian mechanics to Quantum Mechanics.

The framework of Newtonian physics has the following characteristics:

- It is valid for large decoherent systems.
- It is deterministic given the initial conditions and the dynamics.
- Starting from a Hamiltonian or Lagrangian a variational argument gives the equations of motion.
- The position of a particle is a function of time.

In contrast, non-relativistic Quantum Mechanics has the following characteristics:

- It is valid for all slow systems.
- It is probabilistic given the initial conditions and the dynamics.
- The Hamiltonian operator gives the time evolution of a wave function.
- The wave function describes the probabilities for the position of a particle at a point in time.

The transition from Newtonian mechanics to Quantum Mechanics is much more difficult than the transition from Newtonian mechanics to Special Relativity. In Quantum Mechanics, the action S is the integral of the Lagrangian L with respect to time t. If the value S is much larger than $\hbar$, one can use Newtonian mechanics, but also here it makes more sense to use Quantum Mechanics for Particle Physics instead of Newtonian mechanics because it is more general.

Because one treats very small and very fast objects in Particle Physics, one might guess that relativistic Quantum Mechanics would be the framework for Particle Physics. However, this is wrong. The wave function Ψ is normalized in non-relativistic Quantum Mechanics such that $\int |\Psi|^2 d^4x = 1$ where d^4x specifies integration over all four dimensions of spacetime. This just means that the probability to find the particle somewhere in the universe is one. This normalization is in conflict with Special Relativity because in Special Relativity the number of particles in the universe can change. Note, however, that combining a proton and an electron into a hydrogen atom is not what is meant here because this does not change the number of particles. A beta-decay where a neutron decays into a proton, an electron and an antineutrino on the other hand really changes the number of particles.

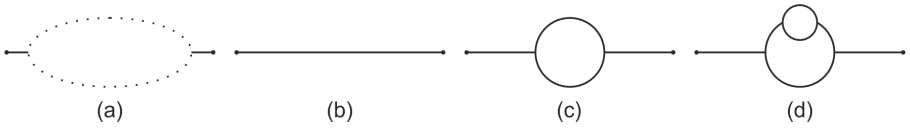


Figure 1.1: Contributions to a moving particle

If one wants to study the probability for a particle starting at the left point in figure 1.1 (a) and ending at the right point, this probability is largely governed by (b) in this figure, but one has to include the possibility (c) that the particle breaks up into two particles which separate, come together, annihilate and create again the initial particle, as well as all the more complicated possibilities such as (d). In Quantum Mechanics one has to allow all possible trajectories from the left point to the right point. This is similar to the double-slit experiment where one has to add all possible paths of the electron. However, here it is not clear how important all these contributions are, and one would like to have a framework where these

[1] As a side remark it can be noted that General Relativity is not a generalization of Special Relativity, and it is also not a framework. Its context is gravitation, and it generalizes Newton's theory of gravity.

quantitative issues are included and handled correctly. The right framework turned out to be Quantum Field Theory which is actually not a theory, but is a very radical sort of revisioning of the starting point and how to work with things.

1.2 A Short Introduction to Quantum Field Theory

A particle can be thought of as moving along a path in spacetime which is also called a world line, and this path in spacetime can be described by some function $x(t)$ as visualized in figure 1.2 (a). The position of the particle in space is parametrized by time. A field is a different picture because it has a field axis in addition to the spacetime axes. One can think of the field as a surface parametrized by space and time as shown in (b) of this figure. The field is therefore a function $\Phi(x,t)$. Particles and fields are very different, but if one considers only small localized fluctuations of the field Φ, and the field is essentially flat everywhere at a value zero with exception of a few blips here and there, these blips are the particles. In (c) of the figure, two such fluctuations representing particles are indicated.

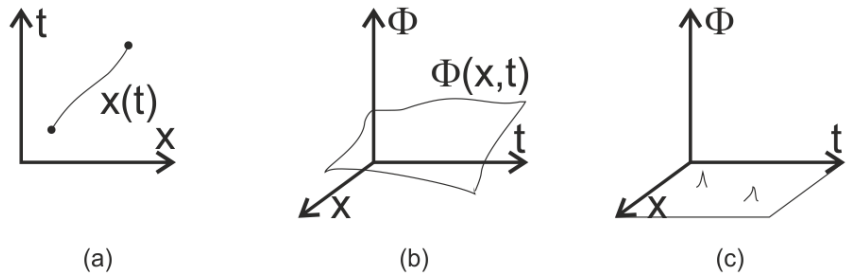


Figure 1.2: Comparison between particle and field

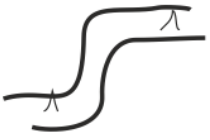
A lake full of water is a good field descriptor. It certainly does not look like a particle, but the little ripples on the surface are more like particles than the whole lake. One has to be careful with this analogy because the lake is made out of water molecules, and these molecules are not the particles meant here. The fields of Quantum Field Theory are not made of anything. In other areas of physics such as density functional theory, mean field theory or the theory of fluids, fields are nice approximate descriptions, but they are always made out of something while the fields in Quantum Field Theory are fundamental.

Quantum field theory has a few nice properties. One is that it gives a natural origin of identical particles. Two electrons in Quantum Mechanics are identical and indistinguishable, and one can see the reason in Quantum Field Theory because two electrons are just two ripples in one field. There is not a field for one electron and another field for the second electron, but there is just one field for all electrons in the universe. Similarly there is one field for every particle such as the muon and tauon, the three flavors of neutrinos, and the six flavors of the quarks.

Quantum field theory also gives weights to diagrams such that it naturally tells how the factor in figure 1.1 (b) compares to the factor of (c), (d) and so on. The coefficients in front of the various terms just pop out of Quantum Field Theory.

A further nice property of Quantum Field Theory is the spin–statistics theorem. A boson is by definition a particle based on Bose statistics which means that it is symmetric under the exchange of two particles. A fermion by definition is based on Fermi-Diract statistics which means that if one interchanges two of them the wave function has to be antisymmetric. The Pauli exclusion principle follows from the Fermi-Dirac statistics. However, the statistics does not have anything to do with spin, but it turns out that every boson has integer spin, and every fermion has half-integer spin. This is a very non-trivial connection whose origin comes from Quantum Field Theory.

Last but not least Quantum Field Theory allows for non-perturbative phenomena. The fluctuations such as the two blips in figure 1.2 (c) are tiny and are also called perturbations, but one can easily imagine fields that have larger structures where the whole field is a kind of waterfall, for example, and one can look for small fluctuations in this waterfall structure. The Higgs mechanism in the Standard Model is an example of such a non-perturbative phenomenon.



1.3 Brief Overview of the Standard Model

One can break up all the fundamental fields of the Standard Model into three categories. The fields are often called particles, but with Quantum Field Theory as the framework for particles physics, the field is the correct paradigm.

The first category contains the force particles also called mediators or gauge bosons. It can also be split into the three categories with the photon represented by γ for electromagnetism, the $W^{\pm}$ and the Z boson for the weak interaction, and the eight gluons for the strong interaction. Electrodynamics and the weak interaction build together the electroweak interaction, and the theory for the strong interaction is called Quantum Chromondynamics.

The second category contains the fields building matter which are all fermions, and it can also be split into two groups called families. The first family consists of the three generations leptons with the electron-neutrino and the electron, the muon-neutrino and the muon, and the tau-neutrino and the tauon. The second family consists of the three generations quarks with the up and down, the charm and strange, and the top and bottom quark. The electron and its neutrino or the up and down quark, for example, are called different flavors. For each of these field or particle there is an antiparticle. Quarks in addition have one of the three colors red, blue, green.

The third and last category contains only the Higgs field or Higgs boson and plays a different role than the other fields. The Higgs field is responsible for the masses of the other fields.

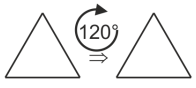
The three categories can also be characterized by spin. The Higgs boson in the third category has spin zero, the matter fields in the second category have spin one half, and the gauge bosons in the first category have spin one. The main question of Particle Physics is how these fields interact with each other. Although the details of the Standard Model are quite nasty, the Standard Model itself is very elegant. It is completely based on symmetry principles. The obviously missing interaction, however, is gravity, and a complete theory or model for Particle Physics must certainly include it.

2 Symmetries and Related Concepts

2.1 Symmetries

The concept of *symmetry* is an incredibly powerful tool to help simplify calculations, but symmetry also plays a more fundamental role in determining the type of dynamics in certain physical theories. For example, Special Relativity is nothing more than a statement about the symmetries of spacetime, and the forces in the Standard Model can be seen to arise as the consequence of certain symmetries. Simply stated, a symmetry is an invariance under some transformation or, in other words, one does something to something, and it does not seem to change.

A symmetry can be *static* such as many objects or shapes in geometry. An equilateral triangle, for example, looks unchanged when rotated by 120° about its center. Static symmetries are easy to visualize, but *dynamical* symmetries such as the transformation of a Lagrangian $L \to L' = L$ will turn out to be important in Particle Physics. Dynamical symmetries arise when looking at the evolution of a system in time.



No matter what type of symmetry is considered, the spirit is the same. A transformation is enacted on something, and afterwards that something looks the same. Naively it may seem that the something must only be built out of things that themselves are invariant. If this were the case it would be terribly restrictive, but fortunately one can build an invariant something out of pieces which are not invariant so long as one combines them in an appropriate way. One can build, for example, a rotationally invariant scalar from vector components with a dot product. The components of the vector change when the coordinate system is rotated, but the length of the vector, which is the dot product of the vector with itself, is invariant.

Thus, symmetries are always based on transformations and invariance under these transformations. In the following the preliminary focus will be on describing transformations in general, but later it is made sure that they are symmetries of a Lagrangian.

2.2 Types of Transformations

Transformations come in different types such as global or local, and to clarify these types static examples will be used as illustrations.

A transformation can either be global or local. A *global* transformation is applied to all element of a system uniformly, while a *local* transformation may be applied to each element differently. In figure 2.1 the transformation in (a) is a translation in space, and the transformation in (b) is rotation in space. If the same translation in space is globally used for all nine dots in (a), the result resembles the original configuration of the nine dots, while the result looks different when each dot it translated differently. The example in (b), however, shows that the result of local and global transformations can look the same because one cannot distinguish whether the four circles were rotated all about the same or each about a different angle. If a system is symmetric under local transformation then it is automatically symmetric under global transformation because one can choose the same transformation for each element of the system instead of different transformations, but the reverse is not true as the examples in figure 2.1 show.

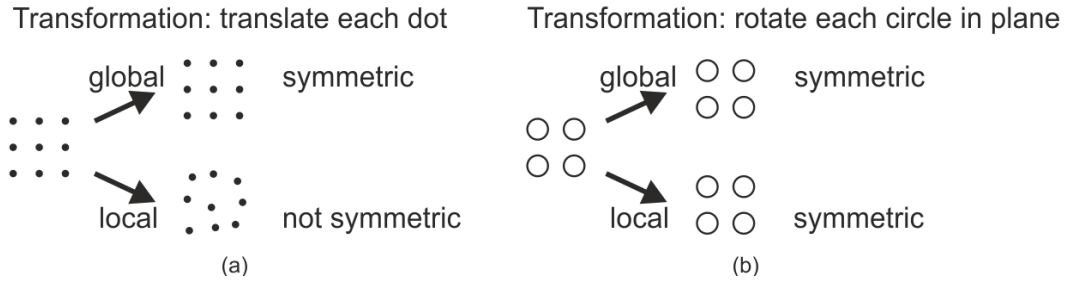


Figure 2.1: Difference between local and global transformations

A transformation can either be *discrete* or *continuous*, a discrete transformation can be either *finite* or *infinite*, and a continuous transformation can either be *compact* or *non-compact*. All four possibilities are illustrated in figure 2.2 with examples leading to a symmetric result. The rotation of the triangle in (a) is discrete and finite because the angle θ can only change in discrete steps and there are only the finite possibilities $\theta \in \{0°, 120°, 240°\}$ creating a symmetry. The global translation of the infinite lattice of equidistant dots in (b) is discrete and infinite because the distance d for the translation T_d must be an integer multiple of the distance between two neighboring dots. The rotation of the circle in (c) is continuous and compact because θ can take any value in $[0, 2\pi)$ and the circle stays for any value of θ in a bound area. The translation of the straight line in (d) is also continuous but non-compact because d in T_d can take any value in $(-\infty, +\infty)$ but it is not bound in the directions to the left and to the right.

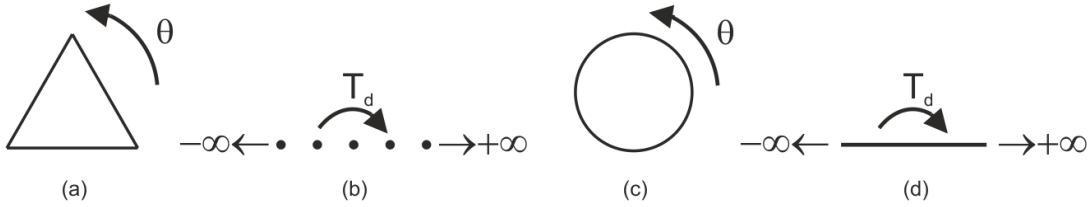


Figure 2.2: Difference between discrete and continuous transformations

Continuous transformations cannot be finite and the distinction between finite and infinite does not make sense. However, the distinction between compact and non-compact plays a similar role as the distinction between finite and infinite for discrete transformations. The distinction between compact and non-compact determines whether something is quantized or not in Quantum Mechanics. Angular momentum is always quantized as it is similar to situation (c) in figure 2.2 while linear momentum is only quantized if the particle is put into a box and it is not quantized in a situation such as (d).

As a last distinction, transformations can be *internal* or in *spacetime* which is also called *external*. The coordinates in spacetime generally change with a spacetime transformation if there is a coordinate system specified for spacetime, while internal transformations do nothing to the coordinates. Special relativity

is associated with spacetime symmetries, but the strong, weak and electromagnetic forces are associated with internal symmetries called gauge symmetries.

Internal transformations in a sense also change coordinates because it turns out that it is very useful to increase the notion of space to include isospace and similar spaces which can also be coordinatized. Thus, one can talk about rotations in these internal spaces. These transformations do not touch on what happens in spacetime and vice versa transformations in spacetime normally do not impact these internal spaces.

2.3 Groups and Representations

For the purpose of Particle Physics, transformations can be treated mathematically using the concepts of groups and representations. A *group* is a collection of elements $\{A, B, ...\}$ with a composition $\bullet$ that satisfies:

1. Closure: If $A, B \in G$ then $A \bullet B \in G$
2. Identity: There is some $I \in G$ such that $I \bullet A = A$ for any $A \in G$
3. Inverse: For any $A \in G$ there is an $A^{-1} \in G$ such that $A^{-1} \bullet A = I$
4. Associativity: $A \bullet (B \bullet C) = (A \bullet B) \bullet C$

The axioms identity and inverse will be very important in building invariants. A group with commutativity $A \bullet B = B \bullet A$ in addition to the above axioms is called *abelian*, but groups needed in Particle Physics do normally not commute and are therefore non-abelian.

A subset of elements of a group that satisfies all axioms of a group is a *subgroup* of the original group. Subgroups always have to include the identity and the inverses for all elements and must also be closed under the composition.

More important than the groups themselves are in physics the representations of the groups. One can abstractly specify a group. The group elements are the transformations, and the composition is the application of two transformations one after the other. The group of two-dimensional rotations, for example, can be defined by the angle for the rotation and the composition as the addition of two angles. In three dimension, rotation becomes much trickier because a rotation, for example, about the x-axis followed by a rotation about the y-axis can be represented by a single rotation about some axis, and one can calculate this axis but it is not as trivial as in the two-dimensional case.

Instead of looking at the group elements, one can look at the objects the group elements act on. The two-dimensional rotations may act, for example, on triangles. These are the *representations* of the group. A single group has usually many different representations. Some are more useful since they fully illustrate the content of the group. They are called *faithful* representations. A group does not only allow different representations, but representations can be used to define the group.

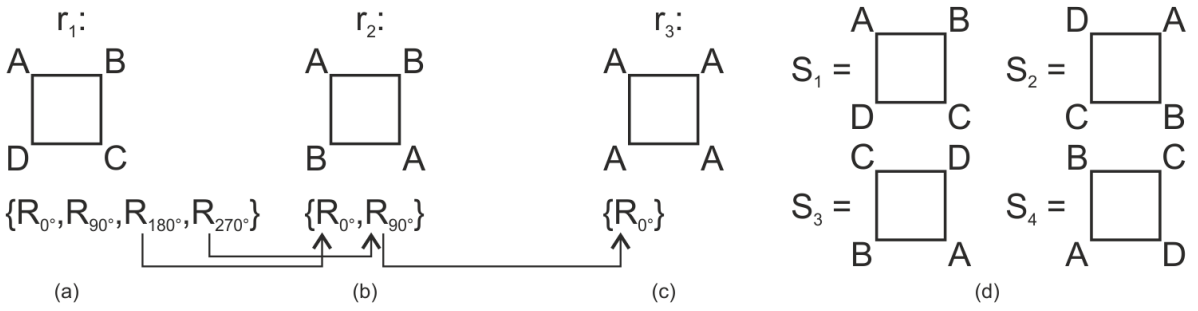


Figure 2.3: Three representations for the group of rotations about 90°

In figure 2.3 three representations of the group of rotations about 90° are shown as an example where the identity is $\mathrm{I} = \mathrm{R}_{0^\circ}$ and the composition is the addition of angles. The representations are squares with letters assigned to the four corners. Representation r_1 in (a) is the only faithful representation because all four group elements R_{0°, R_{90°, R_{180°, and R_{270° are distinguishable through the letters A, B, C, and D assigned to the four corners. The result of the group elements R_{0° and R_{180° as well as those of R_{90° and

R_{270° look the same in (b), and r_2 is a degenerate representation. The result of all four group elements in (c) are indistinguishable such that r_3 is a highly degenerate representation. It is called the identity representation. The four possible squares with distinguishably labeled corners are listed in (d).

It is important to have the option of the identity representation because there are things which do not get touched by certain transformations. Leptons, for example, are not effected by a SU(3) gauge rotation of the strong interaction. Thus, leptons have to be in the identity representation of the SU(3) group.

The simplest group is the one containing only the identity. The simplest non-trivial group has two elements $\{I, g\}$ with the property $g \neq I$ and $g^2 = I$ where g^2 stands for $g \bullet g$. This group is called $\mathbb{Z}_2$ and is abstractly defined. One representation of it contains the two rotations R_0 by 0° and R_π by 180°. A second representation consists of E for even numbers and O for odd numbers with the addition as the composition. A third representation has the two elements $+1$ and -1 with the multiplication as the composition. All three are abelian because their multiplication tables are symmetric with respect to the diagonal from top left to bottom right. For abelian groups the order of the elements in the compositions does not matter, but for non-abelian groups it does. As a convention for multiplication tables the row is the first element and the column the second.

	R_0	R_π
R_0	R_0	R_π
R_π	R_π	R_0

	E	O
E	E	O
O	O	E

	+1	−1
+1	+1	−1
−1	−1	+1

The representations in figure 2.3 use geometrical objects, but one can often work with more convenient *linear* representations where the transformations act linearly using matrices. In the example of the group of rotations by multiples of 90° the definitions

$$S_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad S_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad S_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad S_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$R_{0^\circ} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \quad R_{90^\circ} = \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix} \quad R_{180^\circ} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} \quad R_{270^\circ} = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$$

can be used where a column vector is assigned to each square in (d) and a matrix as an element of the group is assigned to each rotation. These matrices behave as expected under matrix multiplication such that $R_{90^\circ} R_{180^\circ} = R_{270^\circ}$ and so on.

This is a four-dimensional representation, but there are representations of this group with other dimensionalities. This group is an abelian group because rotations in two dimensions can be defined as addition of angles and the addition of numbers is commutative. These matrices have real components, but in Particle Physics also matrices with complex components will be used. Many groups have a matrix representation. This automatically guarantees associativity because matrix multiplication is associative.

2.4 Dual Representations

An important concept is *invariance*. The scalar product of two vectors, for example, is invariant under rotation. The way to describe how elements of a representation transform under an element of a group G has been defined above. To get invariant scalars, one can try to combine two objects which transform oppositely. The opposite transformation for A must be A^{-1}, and the need for a transposition can be understood when looking at the scalar product $a \cdot b$ of two vectors a and b as the matrix multiplication

$$a^T b = (a^1, a^2, a^3) \cdot \begin{pmatrix}b^1\\b^2\\b^3\end{pmatrix}$$

of a row vector a^T with components a^1, a^2, a^3 and a column vector b with components b^1, b^2, b^3.

For any matrix representation r one can form the *dual representation* $\tilde{r}$ such that $\tilde{r} \to (A^{-1})^T$ if $r \to Ar$ for $A \in G$. The value $\tilde{r}^T r$ is invariant because $\tilde{r}^T r \to ((A^{-1})^T \tilde{r})^T A r = \tilde{r}^T ((A^{-1})^T)^T = \tilde{r}^T A^{-1} A r = \tilde{r}^T r$.

If r is an element of a complex matrix representation, the dual representation is also complex, and the invariant is $\tilde{r}^\dagger r = (\tilde{r}^*)^T r$. Because Lagrangians must be real, only real invariants are of interest.

In the example of the rotations by multiples of 90° with

$$R = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \tilde{R} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

where the representation with elements r has the basis R and the dual representation with elements $\tilde{r}$ has the basis $\tilde{R}$, the four possible rotations leave $\tilde{r}^T r$ invariant. With $A = \mathrm{R}_{90^\circ}$ and $A^{-1} = \mathrm{R}_{270^\circ}$

$$\tilde{r}^T r = (e, f, g, h) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = ea + fb + gc + hd \to ((A^{-1})^T \tilde{r})^T A r =$$

$$\left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right]^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \left[\begin{pmatrix} f \\ g \\ h \\ e \end{pmatrix} \right]^T \begin{pmatrix} b \\ c \\ d \\ a \end{pmatrix} = fb + gc + hd + ea = \tilde{r}^T r$$

shows the invariance. Note that in this example $A^{-1} = A^T$.

2.5 Metric

For a group G and a linear matrix representation r one can find an object g called *metric* such that $\tilde{r} = gr$. In other words, one starts with a group and a representation for this group with elements r and finds all elements $\tilde{r}$ of the dual representation by applying g systematically to all elements r. The object g can be and will always be represented by a symmetric square matrix, and it satisfies therefore the equation $\tilde{r}^T r = (gr)^T r = r^T g^T r = r^T g r$. The condition for $r^T g r$ to be invariant under the transformation $r \to Ar$ is

$$A^T g A = g \tag{2.1}$$

because $r^T g r = (Ar)^T g A r = r^T A^T g A r$ must be satisfied.

If the metric g is not known, but the group G such as the group all rotations in three dimensions, for example, is given, one can find the metric g such that the condition (2.1) is satisfied for any A in the group. Thus, one can use the transformations A to figure out what the metric g has to be.

More common is the case where the metric is known. If there is some representation r of a group G, then forming a dual representation $\tilde{r}$ with the known metric g will give an invariant $\tilde{r}^T r$ if $A^T g A = g$ for $A \in G$ as stated in (2.1). This also means that given a representation r and a metric g, one can use the condition (2.1) to find the transformations A which leave $\tilde{r}^T r$ invariant. This is the typical approach in physics to encounter symmetries. One starts with a configuration of particles, fields, dynamical quantities and so on all of which form some representation and finds a set of transformations that are symmetries of $\tilde{r}^T r$ using the metric g.

As an example, vectors v in three dimensions and the metric g are used given by

$$v = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \tilde{v} = gv = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

where the metric is simply the identity I. The example is so trivial that the dual vectors look identical to original vectors. (However, this is not normally the case.) Thus, $\tilde{v}v = v^T g v$ is invariant under any transformation A that satisfies (2.1) and consequently also satisfies $A^T A = I$. This is the orthogonality condition, and the group is called O(3) for three-dimensional orthogonal group. Rotations R in three dimensions form a compact, continuous, non-abelian group, satisfy $R^T R = I$ and belong to O(3).

Writing $R_{x(\theta)}$ for a rotation about the x-axis by an angle θ, one can consider the two matrices

$$R_{x(\theta)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \qquad R'_{x(\theta)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & -\cos(\theta) \end{pmatrix}$$

both satisfying $R^T R = I$. However, only $R_{x(\theta)}$ is a rotation, but $R'_{x(\theta)}$ is not. One difference between the two matrices is that $\det R_{x(\theta)} = +1$ and $\det R'_{x(\theta)} = -1$.

To get only the rotations one can build the so-called special three-dimensional orthogonal group SO(3) which is a subgroup of O(3). It consists of the elements R of O(3) with $\det R = +1$. The proof that SO(3) is a subgroup of O(3) is easy. It is closed under the matrix multiplication because $\det(AB) = \det A \cdot \det B$, the identity exists because $\det I = +1$, the inverse satisfies $\det\left(R^{-1}\right) = +1$ because from $R^T R = I$ follows $R^{-1} = R^T$ and from $\det R^T \cdot \det R = +1$ and $\det(R) = +1$ follows that $\det R^T$ must be positive too, and, finally, associativity is guaranteed because matrix multiplication is associative.

The remaining question is what has been taken away from O(3) to end up with SO(3). The two matrices

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -I \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P^2$$

also build a subgroup of O(3), a discrete subgroup. It is again the simplest non-trivial group $\mathbb{Z}_2$ introduced above. The element P is not a rotation, but reflects x, y, z simultaneously, and it is called a parity transformation on all three coordinates. The group O(3) can be decomposed into the two subgroups SO(3) and $\mathbb{Z}_2$ such that $\mathrm{O}(3) = \mathrm{SO}(3) \times \mathbb{Z}_2$. In other words, all elements of O(3) either belong to SO(3) or can be built by multiplying an element of SO(3) with P. (Note that $-I$ in two dimensions is the rotation about 180° and not a parity transformation, but reflecting only one axis gives a parity transformation. In general, reflecting an odd number of axes gives a parity transformation and reflecting an even number of axes gives a rotation in any dimension.)

Another example is based on complex two-dimensional vectors and the metric $g = I$. Given two column vectors u and v with the complex components u^1, u^2 and v^1, v^2, respectively, the scalar $\tilde{u}^\dagger v$ is invariant under the transformations by 2×2 complex matrices A provided $A^\dagger A = (A^T)^* A = I$. This condition is the unitarity condition and defines the unitary group U(2). Restricting transformations in U(2) to continuous transformations means, similarly to the case of O(3) and SO(3), $\det A = +1$. The subgroup of U(2) with $\det A = +1$ is the special unitary group SU(2).

As a 3×3 matrix, an element of SO(3) has nine components, but one cannot choose them independently. The number of free parameters (or generators as will be discussed later) of the group can be determined by expanding

$$R = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad R^T R = I = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and counting the independent equations. Within the nine equations from $R^T R = I$

$$\begin{array}{lll} (1) : a^2 + b^2 + c^2 = 1 & (2) : ad + be + cf = 0 & (3) : ag + bh + ci = 0 \\ (4) : da + eb + fc = 0 & (5) : d^2 + e^2 + f^2 = 1 & (6) : dg + eh + fi = 0 \\ (7) : ga + hb + ic = 0 & (8) : gd + he + if = 0 & (9) : g^2 + h^2 + i^2 = 1 \end{array}$$

those with numbers (2) and (4), (3) and (7), as well as (6) and (8), respectively, are the same and therefore dependent. The nine coefficients with the six remaining independent equations or conditions leave three free parameters.

The question remains whether the condition $\det A = +1$ is an additional constraint reducing the number of free parameters by one. The answer is no because $\det\left(R^T R\right) = \det R^T \cdot \det R = (\det R)^2 = \det I = +1$ following from $R^T R = I$ has two discrete solutions $\det R = \pm 1$ but it is not possible to reach the case $\det R = -1$ by continuously changing the three parameters.

The group SO(N) of rotations in dimension N has $\frac{1}{2}N(N-1)$ free parameters. In the case of $N = 2$, this means that SO(2) has one free parameter, and in the case of $N = 3$, the number of free parameters of SO(3) is 3 as shown above. Rotations in three dimensions can be seen either as rotations about the three axes x, y, z, or, more correct, as the rotations in the planes x-y, x-z, y-z. In four dimensions SO(4) has 6 free parameters corresponding to the rotations in the planes w-x, w-y, w-z, x-y, x-z, y-z.

This is not the situation in Special Relativity because Special Relativity has the metric written in one of the two forms

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.2}$$

where the left version is more common in General Relativity and the right version is mainly used in Particle Physics. The metric is no longer the identity because the time component has a different sign. The value of $r^T g r$ is invariant under the transformation Λ for a four-vector r according to (2.1) if $\Lambda^T g \Lambda = g$. The group of transformations Λ with $\Lambda^T g \Lambda = g$ and $\det \Lambda = +1$ is the Lorentz group also called SO(1,3).

The determination of the number of free parameters for SU(N) is shown for the example $N = 2$. The complex 2×2 matrix U satisfies

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad U^\dagger U = I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and therefore the equations

$$(1) : aa^* + bb^* = 1 \qquad (2) : ac^* + bd^* = 0 \qquad (3) : ca^* + db^* = 0 \qquad (4) : cc^* + dd^* = 1$$

where (1) and (4) are one equation each because every term is real, and (2) and (3) lead to the same two real equations. The four complex or eight real equations reduce to four real independent equations, but the four free parameters are reduced by one because of $\det\left(U^\dagger U\right) = \det I$. In this case $(\det U^*) \det U = +1$ has not two discrete solutions as in the case of SO(3) but has the continuous set $\det U = e^{i\theta}$ of complex solutions. Thus, $\det U = +1$ creates another non-trivial equation to be satisfied, and the number of independent free parameters reduces to 3.

In general, SU(N) has $N^2 - 1$ free parameters, and U(N) has N^2 free parameters. As will be discussed in the following, U(1) corresponds to the electromagnetic force with 1 gauge boson (photon), SU(2) corresponds to the weak interaction with 3 gauge bosons ($W^\pm$, Z^0), and SU(3) corresponds to the strong interaction with 8 gauge bosons (gluons). The correspondence of the electromagnetic and the weak force is not as simple as it seems here, but this will also become clear in the following.

To summarize, the number of continuous parameters for the orthogonal and unitary groups are

$$\text{SO(N)} : \tfrac{1}{2}N(N-1) \qquad \text{SO(N-n,n)} : \tfrac{1}{2}N(N-1) \qquad \text{U(N)} : N^2 \qquad \text{SU(N)} : N^2 - 1 \tag{2.3}$$

as shown above. For the Standard Model, the groups SO(1,3) for Special Relativity, as well as U(1) with one, SU(2) with three, and SU(3) with eight gauge bosons are of primary interest. The Lagrangian for the Standard Model will be invariant under all four groups. That means that every single ingredient of the Lagrangian has to be in some, but not necessarily the same representation of each of these groups. For example, a quark is a spinor of SO(1,3), a vector of SU(3), a vector of SU(2), a vector of U(1), and the W^+ is a vector of SO(1,3), a singlet of SU(3), an adjoint of SU(2), and a vector of U(1).

Being a vector means that the particle is charged, and being a singlet means that the particle is not touched by the group and therefore gets transformed by the identity transformation of the group. These terms and the terms spinor and adjoint will be introduced detailed later. Vectors of U(1), SU(2) and SU(3) are not vectors in spacetime but in some abstract vector space.

The three groups U(1), SU(2) and SU(3) corresponding to three internal gauge symmetries give rise to the forces and are going to play therefore a very different role than SO(1,3). The spacetime symmetries are handled by the SO(1,3) group, and the Lagrangian has to be invariant under the transformations of this group.

3 Special Relativity

3.1 The Spacetime Group and Lorentz Transformations

Special relativity is the calculational framework for Particle Physics. One does not worry about curvature and gravitational effects from General Relativity. The spacetime group is SO(1,3) with one of the metrics in (2.2). In the following the left metric will be used. A transformation $\Lambda \in \mathrm{SO}(1,3)$ satisfies $\Lambda^T g \Lambda = g$ and $\det \Lambda = +1$. The six free parameters can be thought of as rotations in a general sense where the three rotations R_{xy}, R_{xz}, R_{yz} in the planes x-y, x-z, y-z, respectively, are the real rotations in the three-dimensional space, and the remaining three rotations B_{xt}, B_{yt}, B_{zt} in the planes t-x, t-y, t-z, respectively, are the boosts with some velocity in the direction of x, y, z. But instead of calling these transformations R_{xy} or B_{xt}, they are called Λ_{xy} or Λ_{xt}.

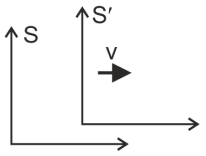
The relevant definitions for the infinitesimal coordinate displacement $d\vec{s}$, a rotation in the y-z plane and a boost in the direction x are

$$d\vec{s} = \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} \qquad \Lambda_{yz}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \qquad \Lambda_{xt}(\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

where θ is the rotation angle, and $\beta = \frac{v}{c}$ together with $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ defines the boost.

According to the definition of the metric, the scalar $d\vec{s}^T g\, d\vec{s} = -c^2dt^2 + dx^2 + dy^2 + dz^2$ is invariant under the transformation $d\vec{s} \to \Lambda d\vec{s}$ if $\Lambda^T g \Lambda = g$. One can easily check this for $\Lambda_{yz}(\theta)$ using $\sin^2 x + \cos^2 x = 1$ and less easily for $\Lambda_{xt}(\beta)$ using $-\sinh^2 x + \cosh^2 x = 1$ because the rotation in the y-z plane leaves $dy^2 + dz^2$ invariant and the boost along x leaves $-c^2dt^2 + dx^2$ invariant.

The relation to sinh and cosh is not that obvious. To get a physical picture one can consider the spatial origin in a frame S where $dx = dy = dz = 0$ and $cdt \neq 0$. The transformation to the frame S' gives



$$\begin{pmatrix} cdt \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma cdt \\ -\beta\gamma cdt \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{pmatrix}$$

and dx' in the frame S' can be written as $dx' = -\beta\gamma cdt = -\beta cdt' = -vdt'$ or $\frac{dx'}{dt'} = -v$. This is the velocity of the origin of the frame S seen in the frame S'. Thus, an observer in the frame S' which is moving along the x-axis with the speed v sees the origin of the frame S moving along the negative x-axis with the speed v.

3.2 Tensors

With vectors r and dual vectors $\tilde{r}^T$ the scalar $\tilde{r}^T r$ is invariant where $\tilde{r} = gr$. If v^μ is a vector and v_μ a dual vector then these quantities are

$$v^\mu = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \qquad v_\mu = (v_0, v_1, v_2, v_3) = (g_{\mu\nu} v^\nu)^T = \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \right]^T = (-v^0, v^1, v^2, v^3)$$

in the usual notation where $v^0 = ct$, $v^1 = x$, $v^2 = y$, $v^3 = z$, and with the Einstein summation convention which says that any two repeated indices are summed. The value

$$v_\mu v^\mu = v_0 v^0 + v_1 v^1 + v_2 v^2 + v_3 v^3 = (-v^0, v^1, v^2, v^3) \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2$$

is invariant because r transforms as $r \to \Lambda r$ and $\tilde{r}$ transforms as $\tilde{r} \to (\Lambda^{-1})^T \tilde{r}$. Written using indices the transformation is for vectors $v^\mu \to v^{\mu'} = \Lambda^{\mu'}_{\mu} v^\mu = \Lambda v^\mu$ and for dual vectors $v_\mu \to v_{\mu'} = v_\mu \Lambda^{\mu}_{\mu'} = v_\mu \Lambda^{-1}$ such that $\Lambda^{\mu}_{\mu'} = (\Lambda^{\mu'}_{\mu})^{-1}$ because $v_{\mu'} = (\tilde{r}^T)' = (\tilde{r}')^T = ((\Lambda^{-1})^T \tilde{r})^T = \tilde{r}^T \Lambda^{-1} = v_\mu \Lambda^{-1}$. An example of a vector is the infinitesimal displacement $d\vec{s} = dx^\mu$ in (3.1) with $dx_\mu dx^\mu = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = ds^2$.

Objects like v^μ, v_μ or Λ^μ_ν can be expressed as matrices but many quantities in physics cannot. The more general mathematical objects needed are so-called *tensors*. One can think of a vector as an object that transforms as $v^\mu \to v^{\mu'} = \Lambda^{\mu'}_{\mu} v^\mu$ and a dual vector as an object that transforms as $v_\mu \to v_{\mu'} = \Lambda^{\mu}_{\mu'} v_\mu$. (In matrix notation the order must be $v_\mu \Lambda^{\mu}_{\mu'}$ as used above, but in tensor notation with the Einstein summation convention the order is not relevant.)

One can define tensors with arbitrary rank (p, q):

(0,0)	scalar	$T \to T' = T$	$T \to T' = T$
(1,0)	vector	$T^\mu \to T^{\mu'} = \Lambda^{\mu'}_{\ \nu} T^\nu$	$T \to T' = \Lambda\, T$
(0,1)	dual vector	$T_\mu \to T_{\mu'} = \Lambda^{\nu}_{\ \mu'} T_\nu$	$T \to T' = T\, \Lambda^{-1}$
(1,1)	(1,1) tensor	$T^\mu_{\ \nu} \to T^{\mu'}_{\ \nu'} = \Lambda^{\mu'}_{\ \kappa} \Lambda^{\lambda}_{\ \nu'} T^\kappa_{\ \lambda}$	$T \to T' = \Lambda\, T\, \Lambda^{-1}$
(2,0)	(2,0) tensor	$T^{\mu\nu} \to T^{\mu'\nu'} = \Lambda^{\mu'}_{\ \kappa} \Lambda^{\nu'}_{\ \lambda} T^{\kappa\lambda}$	$T \to T' = \Lambda\, T\, \Lambda^T$
(0,2)	(0,2) tensor	$T_{\mu\nu} \to T_{\mu'\nu'} = \Lambda^{\kappa}_{\ \mu'} \Lambda^{\lambda}_{\ \nu'} T_{\kappa\lambda}$	$T \to T' = (\Lambda^{-1})^T\, T\, \Lambda^{-1}$

...

This table shows the rank, the name, the transformation in index notation, and the transformation in matrix notation. The index notation is more useful than relying on matrix manipulation because the order does not matter, one can express higher rank tensors, equations become simpler, and transformation properties are more transparent. Note, however, that $\Lambda^\kappa_{\ \mu} \Lambda^\lambda_{\ \nu}$, for example, and similar for higher ranks does not represent two separate transformations but is one single transformation.

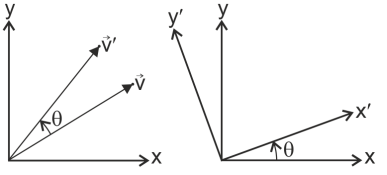
The rank (p, q) indicates the number p of upper and the number q of lower indices where the distinction between upper and lower indices is the distinction between a representation and a dual representation of a group. In Special Relativity, the group is SO(1,3), and the representation r leads to upper indices and the dual representation $\tilde{r}$ to lower indices. Tensors can be combined with index contraction as in $G^\nu = v_\mu T^{\mu\nu}$ or without as in $H_\mu^{\ \lambda\nu} = v_\mu T^{\lambda\nu}$. Some tensors can be represented by matrices, but many cannot. The indices essentially tell how a tensor transforms. Each upper index transforms like a vector, and each lower index transforms like a dual vector.

In Special Relativity, three categories of quantities may appear in index notation with upper and lower indices:

1. Tensors: The kinematic and dynamical quantities will almost always be represented by some tensor. They represent the degrees of freedom. Velocity and the electromagnetic field strength, for example, are written as tensors.
2. Metric: The metric tensor $g_{\mu\nu}$ takes an element of the representation r to an element of the dual representation $\tilde{r}$ and therefore turns a vector into a dual vector. The metric has the special property $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. The inverse metric $g^{\mu\nu}$ does the opposite and turns a dual vector into a vector. The metric is a true tensor and transforms accordingly, and it is also a dynamical field in General Relativity, but is constant in Special Relativity. However, $\Lambda^T g \Lambda = g$ and therefore $g = (\Lambda^{-1})^T g \Lambda^{-1}$ or $g = g'$. In Special Relativity, the metric is often called $\eta_{\mu\nu}$, and it satisfies $\eta^{\mu\nu} = \eta_{\mu\nu}$.
3. Transformations: The transformations operate on tensors, can always be represented by a matrix, and always carry one index from the old coordinates and one from the new. Transformations are different from tensors, and one never transforms transformations.

The difference between matrix notation and index notation can be demonstrated with the example of the two-dimensional rotations. The two matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$



both define a rotation about the angle θ. The left rotation is called active and transforms a vector $\vec{v}$ into a vector $\vec{v}'$ by rotating it about the angle θ. The right rotation is called passive and changes the coordinate system with coordinates x and y into the coordinate system x'

and y' by rotating it about the angle θ. Usually in physics the coordinate transformation is meant when talking about a rotation. The rotation in matrix notation

$$dx^\mu = \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \rightarrow \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta\, dx + \sin\theta\, dy \\ -\sin\theta\, dx + \cos\theta\, dy \end{pmatrix} = \begin{pmatrix} dx' \\ dy' \end{pmatrix}$$

needs the matrix on the left side and the vector dx^μ on the right side. The rotation in index notation, however, with its components

$$\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \Lambda^{1'}{}_{1} = \cos\theta \qquad \Lambda^{1'}{}_{2} = \sin\theta \qquad \Lambda^{2'}{}_{1} = -\sin\theta \qquad \Lambda^{2'}{}_{2} = \cos\theta$$

applied to $dx^\mu \rightarrow dx^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, dx^\mu = \Lambda^{\mu'}{}_{1}\, dx^1 + \Lambda^{\mu'}{}_{2}\, dx^2$ corresponding to $dx' = \cos\theta\, dx + \sin\theta\, dy$ and $dy' = -\sin\theta\, dx + \cos\theta\, dy$ is independent of the order such that $dx^\mu\, \Lambda^{\mu'}{}_{\mu}$ gives the same result.

Humans usually prefer matrix multiplication, and multiplication with tensors of rank 2 can be turned into matrix multiplication by making sure that repeated indices are directly next to each other. Thus, $M_{\mu\nu}T^{\lambda\mu}$ needs to be changed into $T^{\lambda\mu}M_{\mu\nu}$ by simply switching the order, and $M_{\mu\nu}T^{\mu\lambda}$ needs to be changed by transposing one of the two matrices. If the Lorentz transformation $\Lambda^{\mu'}{}_{\mu}$ is represented by the matrix Λ, then $\Lambda^{\mu}{}_{\mu'}$ is represented by Λ^{-1}, $\Lambda_{\mu}{}^{\mu'}$ by Λ^T, and $\Lambda_{\mu'}{}^{\mu}$ by $(\Lambda^{-1})^T$.

3.3 Derivatives

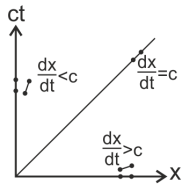
In the three-dimensional space, the quantity $ds^2 = dx_i dx^i = dx^2 + dy^2 + dz^2$ is only zero if all the components dx^i are zero. In the four-dimensional spacetime this is different because of the negative time component. The quantity $ds^2 = dX_\mu dX^\mu = -c^2dt^2 + dx^2 + dy^2 + dz^2$ can be positive, zero or negative. It is a scalar, and it has the same value in all reference frames. It is therefore invariant under Lorentz transformations.

An infinitesimal displacement dX^μ in x-direction has the three possibilities

$$ds^2 = -c^2dt^2 + dx^2 > 0 \qquad \rightarrow \qquad \frac{dx}{dt} > c \qquad \text{spacelike}$$

$$ds^2 = -c^2dt^2 + dx^2 = 0 \qquad \rightarrow \qquad \frac{dx}{dt} = c \qquad \text{lightlike}$$

$$ds^2 = -c^2dt^2 + dx^2 < 0 \qquad \rightarrow \qquad \frac{dx}{dt} < c \qquad \text{timelike}$$

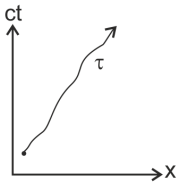


where the first case shows a speed greater than the speed of light, the second case a speed equal to the speed of light, and the third case a speed less than the speed of light. Particles with a mass always belong to the third group and particles without mass to the second group.

The main tools for derivation in non-relativistic physics are either $\vec{\nabla}$ resulting in the gradient $\vec{\nabla}\Phi$ when applied to a scalar field and in the divergence $\vec{\nabla} \cdot \vec{E}$ when applied to a vector field or $\frac{d}{dt}$ which operates on both types of fields. Both of these derivatives need relativistic generalizations.

The generalization of $\vec{\nabla}$ is the 4-derivative $\partial_\mu = \frac{\partial}{\partial X^\mu} = (\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ that acts on fields where X^μ is the position in spacetime. As the index suggests, this is a dual vector $\partial_\mu \rightarrow \partial_{\mu'} = \Lambda^{\mu}{}_{\mu'}\, \partial_\mu$. Similar to the case in three dimensions, the 4-gradient $\partial_\mu\, \Phi$ is a dual vector, and the 4-divergence $\partial_\mu\, A^\mu$ is a scalar. This kind of derivative plays an important role when working with fields.

Experiments, however, are usually working in terms of particles as fluctuations of fields. In non-relativistic physics the position of a particle is written as $\vec{r}(t)$, its velocity as $\vec{v} = \frac{d\vec{r}}{dt}$ and so on. Using $d\vec{s}$ from equation (3.1), the result of $\frac{d\vec{s}}{dt}$ does not transform as a tensor of any kind and is therefore not a valid generalization of $\frac{d}{dt}$ and not a useful definition of 4-velocity. To remedy this situation something that parametrizes the path of a particle and replaces time but is invariant is needed. An obvious solution is the "length" of the path. The invariant displacement $d\tau = \sqrt{-ds^2}$ is used instead of dt.



The scalar $\tau = \int \sqrt{-ds^2} = \int \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$ is called *proper time*, is a monotonically increasing parameter for the path of a massive particle in spacetime and parametrizes the particle's evolution as time does in non-relativistic physics. It is also called "rest" time because $d\tau = cdt$ in the rest frame of the particle where $dx = dy = dz = 0$. The parameter τ for the length of the path cannot be used for massless particles because they move with the speed of light, and in this case $\tau = 0$.

3.4 Velocity, Momentum and Energy

The 4-vector *velocity* defined as $U^\mu = c\frac{dX^\mu}{d\tau}$ is a true tensor that transforms as $U^\mu \to U^{\mu'} = \Lambda^{\mu'}_{\ \mu} U^\mu$. (Note that the factor c gets the units right since $\frac{dX^\mu}{d\tau}$ is dimensionless.) One can calculate how the velocity U^μ of a particle with rest frame S_{rest} looks like when seen from a frame S. Velocity is in components

$$\begin{array}{llll} U^0 = c^2\frac{dt}{d\tau} & U^1 = c\frac{dx}{d\tau} = c\frac{dx}{dt}\frac{dt}{d\tau} & U^2 = c\frac{dy}{d\tau} = c\frac{dy}{dt}\frac{dt}{d\tau} & U^3 = c\frac{dz}{d\tau} = c\frac{dz}{dt}\frac{dt}{d\tau} \\ = \gamma c & = \gamma v_x & = \gamma v_y & = \gamma v_z \end{array} \tag{3.2}$$

because from $d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ follows $\frac{d\tau^2}{dt^2} = c^2 - v^2$, $\frac{dt^2}{d\tau^2} = \frac{1}{c^2 - v^2}$ and $\frac{dt}{d\tau} = \frac{\gamma}{c}$. The 4-vector velocity can therefore be written as

$$U^\mu = \begin{pmatrix} \gamma c \\ \gamma \vec{v} \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

by combining the three spatial components into a three-dimensional vector $\vec{v}$. (Note that all four components depend on all three components of the spatial velocity $\vec{v}$ because of γ.)

The 4-vector *momentum*, which is perhaps the most important quantity for studying collisions including decays, is defined as $P^\mu = mU^\mu$ with the invariant mass m which is sometimes also called rest mass. One can write the momentum as

$$P^\mu = \begin{pmatrix} m\gamma c \\ m\gamma \vec{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{c}E \\ \vec{P} \end{pmatrix} \qquad P^\mu_{\text{rest}} = \begin{pmatrix} mc \\ 0 \end{pmatrix} \tag{3.3}$$

in general and in the rest frame, respectively. For $\frac{v}{c} \ll 1$ one gets $\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$. Applied to (3.3) gives $m\gamma c \approx \frac{1}{c}(mc^2 + \frac{1}{2}mv^2 + ...)$ for the time component and $m\gamma\vec{v} \approx m\vec{v} + ...$ for the space components. The second term in the time component is the non-relativistic kinetic energy, and the first term of the space components is the non-relativistic momentum $\vec{p}$. (The first term of the time component is the well-known $E = mc^2$ in the rest frame.) The 4-vector momentum consists of the relativistic energy E divided by c as the time component and the relativistic momentum $\vec{P}$ as the space components.

Because P^μ is a vector, $P_\mu P^\mu = -(P^0)^2 + (P^1)^2 + (P^2)^2 + (P^3)^2$ is a scalar and must therefore be invariant. The vector P^μ is different for every frame, but $P_\mu P^\mu = -\frac{E^2}{c^2} + P^2$ is the same in every frame, and it is especially the same in the rest frame where it takes the value $P_\mu P^\mu = -m^2c^2$ according to (3.3). The resulting formula usually written as

$$E^2 - \left|\vec{P}\right|^2 c^2 = m^2c^4 \qquad E^2 - P^2c^2 = m^2c^4 \tag{3.4}$$

is called the *mass-shell* condition and relates relativistic energy and momentum to mass. It must be obeyed by all real particles but not necessarily by virtual particles. If $P_\mu P^\mu < 0$ (timelike) then $m^2 > 0$ and the particle is massive. If $P_\mu P^\mu = 0$ (lightlike) then $m^2 = 0$ and the particle is massless. If $P_\mu P^\mu > 0$ (spacelike) then $m^2 < 0$ and the particle is tachyonic.

In collision experiments the total initial 4-momentum P^μ_{initial} must be equal to the total final 4-momentum P^μ_{final}. Because they are vectors, they have to be specified in the same frame when written as an equation $P^\mu_{\text{initial}} = P^\mu_{\text{final}}$. However, if one squares both total momenta they still are the same in the same frame, but as invariants they are also equal in different frames. Quite often in collision problems, one works in the lab frame where one sets up the particles to be shot into an accelerator. On the other hand, one would like to do a lot of calculations in the center of momentum frame where the net spatial momentum is zero and a lot of quantities simplify. Thus, one has to square the 4-momenta.

4 Lie Groups, Lie Algebras and Spinors

4.1 General Comments about Abstraction

Often in mathematics and physics one defines things in the most concrete and intuitive way one can, but sometimes using a more abstract definition allows to extend the applicability of the idea at hand. For example, the first definitions for trigonometric functions $\sin\theta$ and $\cos\theta$ are usually in terms of real rectangular triangles one can physically construct and measure. To measure the values of $\sin\theta$ and $\cos\theta$ where θ is one of the angles less than 90°, one can measure the length of the opposite or adjacent side and divide it by the length of the hypotenuse. However, this definition does not help for $\cos(i\theta)$ with $i = \sqrt{-1}$. The definition of $\cos x$ by its Taylor series $\cos x = 1 - x^2/2! + x^4/4! + ...$ allows to evaluate $\cos(i\theta)$ by setting $x = i\theta$, because this only uses known multiplication of complex numbers.

So far the discussion of rotations and Lorentz transformations has heavily focused on concrete representations like coordinates in space. After all $dX^\mu \to dX^{\mu'} = \Lambda^{\mu'}_{\ \mu}\, dX^\mu$ is defined by how it acts on coordinate differentials. From this basic starting point one can build more general vectors, dual vectors, and higher rank tensors.

One thing that is missing is the possibility for comparing different continuous groups. For discrete groups one can just compare their multiplication tables to see if they are the same. For continuous groups with an infinite number of elements constructing and comparing multiplication tables is impossible. However, for the groups relevant in physics there seem to be only a handful of distinct types of transformations and everything else is built out of them. Rotations in space, for example, can be seen as the three types rotations in the x-y, in the x-z, and in the y-z plane.

For comparing continuous groups, the following properties might be important:

- number of free parameters,
- abelian or non-abelian,
- dimensionality of matrix representations,
- real or complex.

The first two properties turn out to be important, but the other two do not because one single group can have several matrix representations with different dimensions and the group of rotations in two dimensions has real and complex representations.

4.2 Lie Groups

Most groups used in physics are *Lie groups*. This includes the groups SO(1,3), SU(3), SU(2), U(1) as well as purely spatial rotations SO(N) used in the Standard Model. A Lie group is defined as group which forms a manifold with a differentiable structure. What is important about this definition is that if one knows how to do something infinitesimally, then one can figure everything else out, or almost everything.

For example, if one knows how to rotate by 1° in a plane, then one can build any integer rotation in this plane by composing these. A rotation by 45° is 45 rotations by 1° applied one after the other, or if the rotation by 1° is given as the matrix $R(1°)$ then the matrix $R(45°)$ is equal to $R(1°)^{45}$. This does not allow a rotation by 0.5°, but if $R(\varepsilon)$ is given and ε is small enough, one can build a rotation about any angle. If ε is an infinitesimal quantity, however, an infinite number of these infinitesimal rotations is needed to come up with a finite rotation.

For Lie groups there is automatically a built-in mechanism for doing this compounding of infinitesimals. A general element of a Lie group can be written as the exponential map

$$A = e^{ig_A v^A} \tag{4.1}$$

where g_A is said to generate the transformation associated with A and v^A parametrizes it. (Roughly speaking $R(1°)$ is the generator in the above analogy and 45° is the parameter.)

Note that if A is an $N \times N$ matrix, then so is g_A. The expression $g_A v^A$ allows to combine transformations of different types. For example, $g_A v^A = g_{xy}\alpha + g_{xz}\beta + g_{yz}\gamma$ in SO(3) combines a rotation about α in

the x-y plane, a rotation about β in the x-z plane and a rotation about γ in the y-z plane. This is good because $R_{xy}(\alpha)\,R_{xz}(\beta)\,R_{yz}(\gamma)$ must combine to give a single rotation. One should think of g_A as a dual vector of generators and v^A as a vector of parameters.

Rotations in the three-dimensional space as elements of SO(3) can serve as a concrete example. To define the exponential the Taylor expansion of the exponential function

$$R_{yz}(\theta) = e^{ig_{yz}\theta} = I + ig_{yz}\theta + \frac{1}{2!}(ig_{yz}\theta)^2 + \frac{1}{3!}(ig_{yz}\theta)^3 + ...$$

is used where $(ig_{yz}\theta)^2 = -g_{yz}^2\theta^2$ and so on. The generators g_{xy}, g_{xz}, g_{yz} of SO(3) are the 3×3 matrices

$$g_{xy} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad g_{zx} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad g_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{4.2}$$

but are not themselves elements of SO(3). These matrices can be determined from

$$R_{yz}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{1}{2!}\theta^2+... & \theta-\frac{1}{3!}\theta^3+... \\ 0 & -\theta+\frac{1}{3!}\theta^3-... & 1-\frac{1}{2!}\theta^2+... \end{pmatrix} \qquad g_{yz}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $g_{yz}^3 = g_{yz}$. Finding g_{yz} is a bit tricky, but verifying that g_{yz} inserted into the Taylor expansion of $R_{yz}(\theta)$ is rather easy. Thus, one can think about SO(3) in terms of the three infinitesimal generators g_{xy}, g_{yz}, g_{zx}.

The three matrices are not unique, and there are many different sets of three matrices being infinitesimal generators of SO(3). However, also the chosen representation of the rotations is not unique because it is based on the selection of a coordinate system and on the rotations in the three planes spanned by the three axes. The generators form a basis in the tangent space of the group manifold at the origin, so the analogy to coordinates is quite deep.

4.3 Lie Algebras

The discussion so far has been very coordinate dependent. The rotation $R_{yz}(\theta)$ and the three generators are all based on the chosen coordinate system. However, one would like to generalize these ideas in a more abstract way independent of coordinates.

There is a way to define rotations in three dimensions without referencing these matrices. Instead of the generators as matrices in a given coordinate system one can specify the *Lie algebra* of the group. The Lie algebra defines the algebra of the communicators defined as $[g_1, g_2] = g_1g_2 - g_2g_1$ for the generators g_1 and g_2. For the generators of SO(3) one of the commutators is $[g_{yz}, g_{zx}] = g_{yz}g_{zx} - g_{zx}g_{yz} = ig_{xy}$, and the complete Lie algebra for SO(3) can be written as

$$[g_i, g_j] = i\,\epsilon^{ijk}\,g_k \tag{4.3}$$

where ϵ^{ijk} is $+1$ for an even permutation of 123, is -1 for an odd permutation of 123 and is 0 otherwise. (The upper and the lower indices do not matter in this context, and ϵ^{ijk} is not a tensor because it does not change components through coordinate transformations.) In this form, the generators of SO(3) are specified independently of coordinates. A Lie group can be characterized in the vicinity of the identity by the Lie algebra of its generators. The more general equation for Lie algebras is $[g_i, g_j] = f^{ijk}\,g_k$ where one sums over all generators through index k, and where the terms f^{ijk} are called the *structure constants* of the group.

There are two directions to attack this problem. One can start from the rotations with coordinates, determine the generators in (4.2) and find the corresponding Lie algebra as it has been done here. But one can also start by looking for three 3×3 matrices satisfying (4.3), and one set of matrices found this way would be the three generators (4.2). Using them and the exponential map, one could work backwards to the rotations such as $R_{yz}(\theta)$. The generators (4.2) have been derived by looking on how rotations in the three-dimensional space act on vectors and components of tensors. There are different representations of SO(3) whose dimensions are a multiple of 3. The question is whether there is also a two-dimensional representation of the rotations in three dimensions.

4.4 Spinors in Three Dimensions

Starting with the Lie algebra (4.3) one can indeed find the matrices

$$g_{yz} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_x \qquad g_{zx} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\sigma_y \qquad g_{xy} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_z \tag{4.4}$$

which satisfy it and where the matrices $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, $\sigma_z = \sigma_3$ are the Pauli spin matrices.

Using (4.1) the rotation about θ in the y-z plane becomes

$$R_{yz}(\theta) = e^{ig_{yz}\theta} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & i\sin\left(\frac{\theta}{2}\right) \\ i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

and similar for R_{zx} and R_{xy}. These matrices satisfy $U^\dagger U = I$ and $\det U = +1$ and therefore belong to SU(2) which act on complex two-component objects χ called *spinors*. They are two-component objects but reflect rotations in three dimensions. Note that in the following matrix notation is used for spinors and not index notation[2] as for tensors.

In addition to the fact that there are objects called spinors this shows that in some sense SO(3) is the same as SU(2) because they share the exact same Lie algebra such that SO(3) $\sim$ SU(2). They are the same near the identity I, but they are globally different because $R_{yz}(2\pi)$ give the different results

$$R_{yz}(2\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \qquad\qquad R_{yz}(2\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

on the left side for SO(3) and on the right side for SU(2). They both give the same result for $R_{yz}(4\pi) = I$. Thus, SO(3) and SU(2) are both representations for the group of rotations in three dimensions, but SU(2) is more faithful and SO(3) is degenerate similar to the situation in figure 2.3. If only working with tensors one would be blind to the possibility $R_{yz}(2\pi) = -I$, and this is relevant because matter behaves like this.

Some people say that spinors know about the square root of the geometry because they probe geometry more deeply than coordinates and tensors. They do not only satisfy the Lie algebra (4.3) but also the anti-commutator algebra $\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_{2\times 2}$ which is called a Clifford algebra. The identity $I_{2\times 2}$ is a 2×2 matrix in spin space, and δ_{ij} is the metric in $\mathbb{R}^3$, a fact that will come in handy later when dealing with spinors in Special Relativity.

The fact that SU(2) is more faithful than SO(3) can also be illustrated by Quantum Mechanics. If one only has integer spin, then by combining spins one can only ever build more integer spin states. However if one allows half integer spin states, then one can build half and whole integer spin states just using half spin states. Thus, if all one has are integer spin states or tensor states one stays in them, but from half integer states one can build also the integer and tensor states. In this sense spinors know what tensors know but know more.

When discussing tensors an important question was how they transform in order to build invariants needed for the Lagrangian. This question is also important for spinors. If the transformations

$$\chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi \qquad\qquad \tilde{\chi} \to \tilde{\chi}' = \left((e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}})^{-1}\right)^T \tilde{\chi}$$

hold where $\vec{\sigma}$ is the vector of Pauli matrices and $\vec{\theta}$ is the vector of parameters, then $\tilde{\chi}^\dagger\chi$ is invariant. The dual $\tilde{\chi}$ is built out of χ with $\tilde{\chi} = g\chi$ using the metric g for spinors. From $R^\dagger g R = g$ for the metric and from $R^\dagger R = I$ for SU(2) it is clear that the metric is I and $\tilde{\chi} = g\chi = \chi$. Thus, $\chi^\dagger\chi$ is invariant.

It follows that $R^\dagger = (e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}})^\dagger = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} = R^{-1}$. Since $\vec{\theta}$ is real and $i \to -i$ already makes the Hermitian conjugate, the Pauli matrices must be Hermitian such that $\vec{\sigma}^\dagger = \vec{\sigma}$. (It can be made more explicitly by Taylor expanding.) This is not very surprising here and the Pauli matrices are really Hermitian, but for the relativistic case, it will be crucial because it is no longer the case.

[2]There is a formalism called the spinor index notation but is mostly used in supersymmetric Particle Physics and not in ordinary Particle Physics.

4.5 Spinors in Four Dimensions

When going from three-dimensional space to one- plus three-dimensional spacetime, vectors become 4-vectors and rotations become Lorentz transformations which include rotations in space where the time coordinate is not touched but also include boosts. To find the relativistic spinors one has to repeat again the full development path from the generators to the Lie algebra and to spinor representations.

The six generators for Special Relativity corresponding to the three rotations R_{yz}, R_{zx}, R_{xy} and the three boosts B_{xt}, B_{yt}, B_{zt} are usually called J_1, J_2, J_3 and K_1, K_2, K_3. Fortunately the generators corresponding to the rotations are

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.5}$$

as one can guess from the rotations in space. If one takes the various boosts and considers their Taylor expansion, then using the exponential map $B = \exp(iK\delta B)$ gives

$$K_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{4.6}$$

for the generators for the boosts.

The algebra of SO(1,3) becomes

$$[J_i, J_j] = i\,\epsilon^{ijk}\,J_k \qquad [K_i, K_j] = -i\,\epsilon^{ijk}\,J_k \qquad [J_i, K_j] = i\,\epsilon^{ijk}\,K_k \tag{4.7}$$

where the commutators of the generators corresponding to the rotations are taken from (4.3). Unfortunately the rotations and boosts of SO(1,3) do not cleanly split from each other. The rotations build a subgroup, but the boosts do not and the mixed commutators $[J_i, K_j]$ are not zero.

A common trick in physics is to build linear combinations for things that do not nicely separate. Thus, one defines $J_{+i} = \frac{1}{2}(J_i + iK_i)$ and $J_{-i} = \frac{1}{2}(J_i - iK_i)$ and gets the algebra

$$[J_{+i}, J_{+j}] = i\,\epsilon^{ijk}\,J_{+k} \qquad [J_{-i}, J_{-j}] = i\,\epsilon^{ijk}\,J_{-k} \qquad [J_{+i}, J_{-j}] = 0 \tag{4.8}$$

which shows that SO(1,3) $\sim$ SO(3) $\times$ SO(3) at least near the identity. The separation is not between rotations and boosts because the three $J_{\pm i}$ combine rotations and boosts.

For the group SO(3) of rotations in space the corresponding spinors are based on SU(2), and therefore for the group SO(1,3) $\sim$ SO(3) $\times$ SO(3) of Lorentz transformations the corresponding spinors are based on SU(2) $\times$ SU(2). Because each of the two SU(2) acts on complex objects with two components, SU(1,3) respectively SU(2) $\times$ SU(2) acts on complex objects with four components, and these objects are called spinors in four dimensions. The spinors in three dimensions are two-dimensional, and the spinors in four dimensions, independent of whether in four space dimensions or one time and three space dimensions, are four-dimensional. This is unfortunate since vectors and spinors have four components, but the components mean totally different things. For a spinor, the first component has nothing to do with time. Vectors and spinors also have very different transformation properties. This is only a misfortune in four dimensions. In higher dimensions vectors and spinors have a different number of components.

The question is how a spinor χ transform. Rotations in three dimensions with the Pauli matrices satisfy the anti-commutator algebra $\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_{2\times 2}$ for $i, j \in \{1, 2, 3\}$ where $\vec{\sigma}$ is a vector of 2×2 matrices in spin space and $g_{ij} = \delta_{ij}$ is the metric. For Lorentz transformations in spacetime one can guess $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times 4}$ where the 4×4 matrices γ^μ acts on four-component spinors.

For the transformation one might build $\exp\left(\frac{i}{2}\gamma^\mu\theta_\mu\right)$ with $\gamma^\mu\theta_\mu$ replacing $\vec{\sigma}\cdot\vec{\theta}$ but unfortunately this does not work because θ^μ has only four parameters instead of the six parameters needed for the three rotations and the three boosts. The problem is improper notation. In three dimensions labeling rotations by axes instead of planes works fine, but it does not in spacetime. Thus, instead of $(\sigma_1, \sigma_2, \sigma_3)$ one can

think of this equivalently as $(-i[\sigma_2,\sigma_3],-i[\sigma_3,\sigma_1],-i[\sigma_1,\sigma_2])$ using (4.3) and (4.4). Similarly one can use $\sigma^{\mu\nu}=-\frac{i}{4}[\gamma^\mu,\gamma^\nu]$ with the obvious property $\sigma^{\mu\nu}=-\sigma^{\nu\mu}$ together with the angles $\omega_{\mu\nu}$ also with the property $\omega_{\mu\nu}=-\omega_{\nu\mu}$ to parametrize the transformation. This leads to the transformation

$$\begin{aligned}\psi\to\psi'&=e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\,\psi\\ \sigma^{\mu\nu}&=\{\sigma^{01},\sigma^{02},\sigma^{03},\sigma^{23},\sigma^{31},\sigma^{12}\}\\ \omega_{\mu\nu}&=\{\omega_{01},\omega_{02},\omega_{03},\omega_{23},\omega_{31},\omega_{12}\}\end{aligned}\tag{4.9}$$

for the relativistic spinors ψ, and gives $\psi\to\psi'=e^{\frac{i}{4}(\sigma^{23}\omega_{23}+\sigma^{32}\omega_{32})}\,\psi=e^{\frac{i}{2}\sigma^{23}\omega_{23}}\,\psi=e^{\frac{i}{2}\sigma^{23}\theta}\,\psi$ for the example of a rotation by θ in the y-z plane because $\omega_{\mu\nu}=(0,0,0,\theta,0,0)$.

4.6 Dirac Matrices

To find the matrices γ^μ one can start from the Clifford algebra $\{\gamma^\mu,\gamma^\nu\}$. One set of matrices are

$$\gamma^0=-i\begin{pmatrix}0&I\\I&0\end{pmatrix}\qquad\qquad\gamma^i=-i\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix}\tag{4.10}$$

which are 4×4 matrices built with the 2×2 identity I and σ_i matrices defined in (4.3). They have the nice properties:

- $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}I_{4\times4}$ as guessed above
- $(\gamma^0)^2=-I$ and $(\gamma^i)^2=+I$ as a consequence
- $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=0$ or $\gamma^\mu\gamma^\nu=-\gamma^\nu\gamma^\mu$ for $\mu\neq\nu$ because $\eta^{\mu\nu}$ is diagonal

as one can easily verify.

The γ matrices have an off-diagonal block structure, but the generators $\sigma^{\mu\nu}$, for example, for a boost σ^{0i} and similarly for a rotation σ^{ij}

$$\begin{aligned}\sigma^{0i}&=-\frac{i}{4}[\gamma^0,\gamma^i]=-\frac{i}{4}(\gamma^0\gamma^i-\gamma^i\gamma^0)\\&=-\frac{i}{4}\left[(-i)\begin{pmatrix}0&I\\I&0\end{pmatrix}(-i)\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix}-(-i)\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix}(-i)\begin{pmatrix}0&I\\I&0\end{pmatrix}\right]=\frac{i}{2}\begin{pmatrix}\sigma_i&0\\0&-\sigma_i\end{pmatrix}\\\sigma^{ij}&=\frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma_k&0\\0&\sigma_k\end{pmatrix}\end{aligned}\tag{4.11}$$

have a diagonal block structure where each block acts on one of the SU(2) in SO(1,3) $\sim$ SU(2) $\times$ SU(2).

Trying $\tilde\psi=\psi$ for the dual of the four-component spinor ψ to build an invariant $\tilde\psi^\dagger\psi$ similar to the case of the two-component spinors χ in SO(3) $\sim$ SU(2) does not work because in

$$\tilde\psi^\dagger\psi\to(\tilde\psi')^\dagger\psi'=\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\,\psi\right)^\dagger e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\,\psi=\psi^\dagger\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^\dagger\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)\psi\neq\tilde\psi^\dagger\psi$$

only the $(\sigma^{ij})^\dagger=\sigma^{ij}$ are Hermitian but not the $(\sigma^{0i})^\dagger=-\sigma^{0i}$ while in the case SO(3) $\sim$ SU(2) all generators were Hermitian. Thus, it would work for rotations but not for boosts. One can fix this with the choice $\tilde\psi=i\gamma^0\psi$ for the dual and therefore using the metric $g=i\gamma^0$ because

$$\begin{aligned}\tilde\psi^\dagger\psi=(i\gamma^0\psi)^\dagger\psi=-i\psi^\dagger(\gamma^0)^\dagger\psi=i\psi^\dagger\gamma^0\psi\to(\tilde\psi')^\dagger\psi'=i(\psi')^\dagger\gamma^0\psi'=i\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\,\psi\right)^\dagger\gamma^0e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\,\psi\\=i\psi^\dagger\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^\dagger\gamma^0e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\phi=i\psi^\dagger\gamma^0e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi=i\psi^\dagger\gamma^0\psi\end{aligned}$$

using $(\gamma^0)^\dagger=-\gamma^0$ and where the step in the second line of the derivation needs some calculations. Thus, with the definitions

$$g=i\gamma^0\qquad\qquad\tilde\psi=g\psi=i\gamma^0\psi\qquad\qquad\bar\psi\equiv\tilde\psi^\dagger=i\psi^\dagger\gamma^0\tag{4.12}$$

where $\bar\psi$ is called the adjoint, the quantity $\bar\psi\psi$ is invariant.

The γ^μ matrices are matrices in spin space but build a vector in spacetime such that they have the unique property of linking spin space to spacetime. Spin space is tied to the geometry at hand. In fact just as vectors actually live in the tangent space at each point, or tangent bundle over all of spacetime, spinors live in the spin bundle over spacetime. Since both spinors and vectors are tied at some level to spacetime, if one transforms coordinates both get impacted whenever such indices appear.

To simplify notation the transformation (4.9) is written as $\psi \to \psi' = S[\Lambda]\psi$. Thus with

$$S[\Lambda] = e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}} \qquad \psi^a \to \psi^{a'} = S[\Lambda]^{a'}{}_a \psi^a \qquad \bar{\psi}_b \to \bar{\psi}_{b'} = S[\Lambda]^b{}_{b'} \bar{\psi}_b$$

and using indices for spinors, the quantity $\bar{\psi}\psi \to \bar{\psi}'\psi' = \bar{\psi}_b S[\Lambda]^b{}_c S[\Lambda]^c{}_a \psi^a$ with $S[\Lambda]^b{}_c S[\Lambda]^c{}_a = \delta^b{}_a$ is invariant.

In order to figure out how $\gamma^{\mu a}{}_b$ transform one can transform each index such that $\gamma^{\mu a}{}_b \to \gamma^{\mu' a'}{}_{b'}$ where $\mu \to \mu'$ is a Lorentz transformation, and the other two indices are indices for spinors. Thus, the transformation is $\gamma^{\mu a}{}_b \to \gamma^{\mu' a'}{}_{b'} = \Lambda^{\mu'}{}_\mu S[\Lambda]^{a'}{}_a \gamma^{\mu a}{}_b S[\Lambda]^b{}_{b'}$ written ordered ready for matrix multiplication. However, one never has to deal with objects in spin space as such, because only invariants such as $\bar{\psi}\psi$ or $\bar{\psi}\gamma^\mu\psi$ go into a Lagrangian. (Note that $\bar{\psi}\gamma^\mu\psi$ is a vector and is therefore in spacetime and not in spin space.)

5 Systems of Mechanics

5.1 Action Principles and Lagrangians

From the various formalisms describing the dynamics of a physical system including Newtonian, Hamiltonian and Lagrangian mechanics the Lagrangian approach based on an action principle $S = \int L(q, \dot{q})\, dt$ turns out to be most useful. Zero variation $\delta S = 0$ under a deformation of the path with some given boundary conditions leads to the Euler-Lagrange equations of motion $\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) = 0$. This is a good starting point for Particle Physics because one can generalize it from an integral over time dt to a relativistic integral d^4X, from point particles $q(t)$ to fields $\phi(X^\mu)$, and from Classical Mechanics with $\delta S = 0$ to Quantum Mechanics with path integrals. The action S is also a good starting point because it allows naturally to realize symmetries of a system. If one wants to impose rotational symmetry, for example, the Lagrangian needs to be a scalar.

An action S is a functional and not a function. For a function $f(x)$ the maxima and minima can be found by solving $\frac{df}{dx} = 0$. For a functional $S[f(x)]$ one can take the variational derivative $\frac{\delta S}{\delta f(x)}$ with respect to the functions that are its arguments and solving $\frac{\delta S}{\delta f(x)} = 0$ gives the function $f(x)$ which maximizes or minimizes S. This approach will be used in the following for relativistic fields and not for particles because relativistic particles have a lot of issues related to creation and annihilation such that fields turn out to be more convenient. Fields naturally accommodate the creation and annihilation of particles.

The action $S = \int L(q, \dot{q})\, dt$ is replaced by the integral

$$S = \int \mathcal{L}(\Phi, \partial_\mu \Phi)\, d^4X \tag{5.1}$$

with a Lagrangian density $\mathcal{L}$ for a field Φ. Later the integral will be replaced by a path integral, but just looking at the classical equations of motion will reveal useful information. Thus considering a field $\Phi(X^\mu)$ and a region M of spacetime with boundary ∂M and boundary conditions $\Phi(X^\mu|_{\delta M})$ and considering a deformed field configuration $\Phi'(X^\mu) = \Phi(X^\mu) + \delta\Phi(X^\mu)$ with $\Phi'(X^\mu|_{\delta M}) = \Phi(X^\mu|_{\delta M})$ and therefore $\delta\Phi(X^\mu|_{\delta M}) = 0$ leads to the variation of the action

$$\begin{aligned}\delta S &= \int \delta\mathcal{L}\, d^4X = \int \left[\frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\partial_\mu\Phi\right] d^4X \\ &= \int \left[\frac{\partial \mathcal{L}}{\partial \Phi} - \frac{\partial}{\partial X^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\Phi)}\right)\right]\delta\Phi\, d^4X + \int \frac{\partial}{\partial X^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi\right) d^4X = 0\end{aligned}$$

using integration by parts and the fact that partial derivative commutes with variation of the path. Since $\delta\Phi$ is zero on the boundary, the second integral vanishes and can be ignored.

Because this equation should be zero for arbitrary deformation of the path the factor $\delta\Phi$ can be anything and can especially be non-zero. The Euler-Lagrange equations of motion are

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \frac{\partial}{\partial X^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right) = 0 \tag{5.2}$$

for a relativistic field Φ similar to the Euler-Lagrange equations in Classical Mechanics.

The starting point of Lagrangian mechanics is $L = T - V$ fed by $H = T + V$, and there is a lot of machinery that goes with connecting the Hamiltonian and the Lagrangian framework. Thinking of this in slightly different terms results in a Lagrangian $\mathcal{L}_{\text{kinetic}}$ for the kinetic energy and a Lagrangian $\mathcal{L}_{\text{interactions}}$ for the interactions, but one adds them to $\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{interactions}}$ because the sign is not important as long as the content of the added term for the interactions has not been specified.

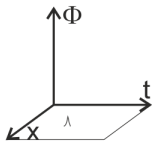
5.2 Equations of Free Motion

A Lagrangian with only a kinetic but no interaction term is called the free-field Lagrangian. The program is to start with the free-field Lagrangian and add the interactions by imposing a symmetry. This is how all the interactions are added in the Standard Model. To write down the analog of $L = \frac{1}{2}mv^2$ for a relativistic field, one needs to consider scalar fields (spin-0, Higgs field), vector fields (spin-1, force mediator particles), and spinor fields (spin-$\frac{1}{2}$, matter particles) with the corresponding Lagrangians

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \tfrac{1}{2}\left(\tfrac{mc}{\hbar}\right)^2\Phi^2 \quad \mathcal{L}_1 = \tfrac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{8\pi}\left(\tfrac{mc}{\hbar}\right)^2 A^\mu A_\mu \quad \mathcal{L}_{\frac{1}{2}} = (\hbar c)\bar\Psi\gamma^\mu\partial_\mu\Psi + mc^2\bar\Psi\Psi \tag{5.3}$$

respectively.

The free Lagrangian for a scalar field Φ is $\mathcal{L}_{\text{free}} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{2}(\frac{mc}{\hbar})^2\Phi^2$ where the first term looks similar to the classical kinetic energy but does not have a factor mass and the second term contains the mass. This also works for massless particles. If the field Φ is zero everywhere except for a small perturbation then this perturbation should have energy even at rest which is proportional to how much the field is excited. Inserting this $\mathcal{L}_{\text{free}}$ into the Euler-Lagrange equations (5.2) gives



$$\partial_\mu\partial^\mu\Phi - \left(\frac{mc}{\hbar}\right)^2\Phi = 0 \tag{5.4}$$

using $\frac{\partial\mathcal{L}}{\partial\Phi} = \left(\frac{mc}{\hbar}\right)^2\Phi$ and $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = \partial^\mu\Phi$. This is the Klein-Gordon equation.

The free Lagrangian for a vector field A^μ is $\mathcal{L}_{\text{free}} = \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\pi}A^\mu A_\mu$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^{\mu\nu} = \eta^{\mu\kappa}\eta^{\nu\lambda}F_{\kappa\lambda}$. With $\frac{\partial\mathcal{L}}{\partial A_\mu} = \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\mu$ and $\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{1}{4\pi}F^{\mu\nu}$ together with the Euler-Lagrange equations (5.2) results in

$$\partial_\mu F^{\mu\nu} - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0 \tag{5.5}$$

which is called the Proca equation. For $m = 0$ two Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ follow. (The other two Maxwell equations require interactions. One can only get two of the four Maxwell equations from an action principle.) This equation would be applicable also in case that the photons have mass.

The free Lagrangian for a spinor field Ψ is $\mathcal{L}_{\text{free}} = (\hbar c)\bar\Psi\gamma^\mu\partial_\mu\Psi + mc^2\bar\Psi\Psi$ where Ψ and $\bar\Psi$ are treated as independent degrees of freedom in the Euler-Lagrange equations (5.2) for reasons to be discussed. Varying with respect to $\bar\Psi$ leads to

$$\not\partial\Psi + \frac{mc}{\hbar}\Psi = 0 \tag{5.6}$$

called the Dirac equation with $\not\partial = \gamma^\mu\partial_\mu$. Varying with respect to Ψ results in $mc^2\bar\Psi - \partial_\mu(\hbar\bar\Psi\gamma^\mu) = 0$ or

$$\not\partial\bar\Psi - \frac{mc}{\hbar}\bar\Psi = 0$$

which is the adjoint of the Dirac equation (5.6). Dirac did not find the equation named after on the path shown here via a Lagrangian and an action principle but by trying to determine the square root of the Klein-Gordon equation (5.4).

Any solution of the Dirac equation (5.6) is – not very surprising as the square root – also a solution of the Klein-Gordon equation (5.4) $\left(\frac{mc}{\hbar}\right)^2 \Psi - \partial_\mu \partial^\mu \Psi = 0$ for each component of Ψ. The γ matrix which is a 4×4 matrix in spin space grabs the components of Ψ in the Dirac equation, mixes them up and relates them to the original components of Ψ. However, in the form of the Klein-Gordon equation there is no γ matrix anymore. One could put in a matrix in spin space but it would be the identity.

Acting with ∂_ν on the Proca equation (5.5) gives $\partial_\nu \partial_\mu F^{\mu\nu} - \left(\frac{mc}{\hbar}\right)^2 \partial_\nu A^\nu = 0$. Because $\partial_\nu \partial_\mu = \partial_\mu \partial_\nu$ is symmetric with respect to swapping indices μ and ν and $F^{\mu\nu}$ is antisymmetric, the term $\partial_\nu \partial_\mu F^{\mu\nu}$ is necessarily zero and this implies $\partial_\nu A^\nu = 0$. The Proca equation with $\partial_\mu({}^\mu A^\nu - \partial^\nu A^\mu) - (\frac{mc}{\hbar})^2 A^\nu = 0$ becomes $\partial_\mu \partial^\mu A^\nu - (\frac{mc}{\hbar})^2 A^\nu = 0$ because $\partial_\mu \partial^\nu A^\mu = \partial^\nu \partial_\mu A^\mu = 0$, and this is again the Klein-Gordon equation (5.4) for each component of A^μ.

Starting from equations (3.3) and (3.4) in the form $P_\mu P^\mu + m^2 c^2 = 0$, replacing P_μ with the operator $i\hbar\partial_\mu$ as usual in Quantum Mechanics and letting it act on Φ, the result is $\partial_\mu \partial^\mu \Phi - (\frac{mc}{\hbar})^2 \Phi$. This is the Klein-Gordon equation. In other words, the Klein-Gordon equation is really just reflecting the mass-shell condition (3.4) which all real degrees of freedom must satisfy[3].

5.3 Degrees of Freedom

To determine the degrees of freedom for particles one can either take the dimension of the configuration space or half the dimension of the phase space. For fields the counting is more subtle. Technically a field has an infinite number of degrees of freedom because one has to specify the field value at each point in spacetime. However the motion of field fluctuations representing particles is covered by the Klein-Gordon equation such that one can count the degrees of freedom per point left over. This approach is called Wigner's classification or the method of induced representations.

One starts with a given momentum P^μ and finds what can be changed without changing the this momentum because the momentum is already covered by the Klein-Gordon equation. The choice of the momentum should not influence the number of degrees of freedom, and one can chose the rest frame to do the counting. Thus P^μ has only the time component which is non-zero. From the transformations of SO(1,3) only the rotations belonging to the so-called little group SO(3) of SO(1,3) do not change the momentum. (It is not obvious, but the degrees of freedom left over in any other frame is the same.) For spin-0 scalars the Klein-Gordon equation does it all and there are no more degrees of freedom left.

For fluctuations of the field representing particles with a non-trivial spin a review of some essential results for angular momentum in three-dimensional non-relativistic Quantum Mechanics will give some insight because only the little group is relevant here. Angular momentum is always quantized. Orbital angular momentum L is just the quantum mechanical version of classical angular momentum. Spin S on the other hand behaves very similar to orbital angular momentum but cannot be give a spacetime picture. It is an internal property of the particle and its total magnitude cannot be changed as the orbital angular momentum can be changed. The value L^2 can be changed, but S^2 is fixed forever. A spin-$\frac{1}{2}$ particle is always a spin-$\frac{1}{2}$ particle. Orbital angular momentum takes integer values, but spin can take half integer values as the Stern-Gerlach experiment shows. If one measures the spin in the z-direction, the outcome for a spin-0 particle is always $S_z = 0$, but can be $-\frac{1}{2}$ or $+\frac{1}{2}$ for a spin-$\frac{1}{2}$ particle and -1, 0 or $+1$ for a spin-1 particle. The degrees of freedom for a spin-$\frac{1}{2}$ particle is 2 for how much is spin up and how much is spin down, and the degrees of freedom for a spin-1 particle is 3. The number of degrees of freedom is the dimension of the corresponding Hilbert space. Because $S^2 = S(S+1)\hbar^2$ the inequality $S^2 > S_z^2 = S^2\hbar^2$ must hold. The reason is that all the spin would be along the z-axis if $S^2 = S_z^2$, and all the three spin components with $S_x = S_y = 0$ would be known simultaneously.

This gives a sense for how many degrees of freedom one should expect. For a scalar field, there should be nothing left because the Klein-Gordon equation is all there is. For the case of the spinor field there should be two degrees of freedom left in addition to momentum covered by the Klein-Gordon equation, and for the vector field there should be three additional degrees of freedom. However, the spinor field as a vector in spin space with four complex components seems to have eight and not three additional degrees of freedom, and the vector field with four real components seems to have four and not three additional

[3]Similarly if one starts with $\frac{p^2}{2m} + V = E$ non-relativistically and replaces $\vec{p}$ by $-i\hbar\vec{\nabla}$ and E by $i\hbar\frac{\partial}{\partial t}$, then one gets the time-dependent Schrödinger equation.

degrees of freedom. The Dirac and the Proca equations are not just the Klein-Gordon equations in a different form but contain more information. These additional constraints reduce the actual number of degrees of freedom to the number of degrees of freedom expected from Wigner's classification scheme.

In the case of the Proca equation (5.5), one can go to the momentum space via $i\hbar\partial_\mu \to P_\mu$ respectively $\partial_\mu \to -\frac{i}{\hbar}P_\mu$ to get

$$-\frac{i}{\hbar}P_\mu\left(-\frac{i}{\hbar}P^\mu A^\nu + \frac{i}{\hbar}P^\nu A^\mu\right) - \left(\frac{mc}{\hbar}\right)^2 A^\nu = -\frac{1}{\hbar^2}P_\mu\left(P^\mu A^\nu - P^\nu A^\mu\right) - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$$

and with $P_\mu = (-mc, \vec{0})$ in the rest frame

$$\frac{1}{\hbar^2}(mc)^2 A^0 + \frac{1}{\hbar^2}(mc)^2 A^0 - \left(\frac{mc}{\hbar}\right)^2 A^0 = 0 \qquad \frac{1}{\hbar^2}(mc)^2 A^i - \left(\frac{mc}{\hbar}\right)^2 A^i = 0$$

for the time coordinate $\nu = 0$ and any of the space coordinates $\nu = i$. From the equation for $\nu = 0$ it follows that A^0 must be zero while the equation for $\nu = i$ is trivially satisfied. Thus there are three values of A^i to be selected corresponding to three degrees of freedom.

In the case of the Dirac equation (5.6), the equation in momentum space becomes $-i\gamma^\mu P_\mu \Psi + mc\Psi = 0$ or $i\gamma^0 mc\Psi + mc\Psi = 0$ in the rest frame where $P_\mu = (-mc, 0, 0, 0)$. Thus $(i\gamma^0 + I)\Psi = 0$ or

$$\left(i\left(-i\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\right) + \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)\begin{pmatrix}\Psi_1\\\Psi_2\\\Psi_3\\\Psi_4\end{pmatrix} = \begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\\Psi_2\\\Psi_3\\\Psi_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

using (4.10). From this follows $\Psi_1 = -\Psi_3$ and $\Psi_2 = -\Psi_4$ leaving two complex or four real degrees of freedom. Therefore Wigner's classification scheme seems to fails. This is however not a failure of the Dirac equation but the wisdom of it because it knows about antimatter. The Dirac equation describes the electron and the positron, each of which is a spin-$\frac{1}{2}$ particle with two degrees of freedom in addition to the part covered by the Klein-Gordon equation.

5.4 Solutions to the Dirac Equation

To find solutions for the Dirac equation (5.6) one can work in the rest frame where $\frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial y} = \frac{\partial\Psi}{\partial z} = 0$ and the Dirac equation reduces to $\gamma^0\partial_0\Psi + \frac{mc}{\hbar}\Psi = 0$. This equation can be written as

$$-\frac{i}{c}\frac{\partial}{\partial t}\Psi_3 + \frac{mc}{\hbar}\Psi_1 = 0 \quad -\frac{i}{c}\frac{\partial}{\partial t}\Psi_4 + \frac{mc}{\hbar}\Psi_2 = 0 \quad -\frac{i}{c}\frac{\partial}{\partial t}\Psi_1 + \frac{mc}{\hbar}\Psi_3 = 0 \quad -\frac{i}{c}\frac{\partial}{\partial t}\Psi_2 + \frac{mc}{\hbar}\Psi_4 = 0$$

in components. The four solutions

$$\begin{aligned}
\Psi^{(1)}_{\text{rest}} &= e^{-i\frac{mc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} & \Psi^{(2)}_{\text{rest}} &= e^{-i\frac{mc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\\
\Psi^{(3)}_{\text{rest}} &= e^{i\frac{mc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} & \Psi^{(4)}_{\text{rest}} &= e^{i\frac{mc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}
\end{aligned} \tag{5.7}$$

solve these equations. From Quantum Mechanics one expects the time dependence of a state to evolve as $e^{-i(E/\hbar)t}$, and for particles at rest the energy is $E = mc^2$. Thus the first two solutions show the usual behavior, but the second two solutions do not.

The second two solutions describe antimatter, and there are different ways to interpret that. One can think of the missing minus sign as negative energy[4], but Feynman and Stückelberg proposed that the

[4]Dirac interpreted the negative energy as a sea of filled states. To create a particle one has to pull a particle out of this sea, and the electron is this particle while the empty space left behind is the positron as the antiparticle of the electron.

corresponding particle has positive energy and moves backwards in time. Antimatter moving backwards in time is the much more natural interpretation than the negative energy assumption, and it will turn out to be useful in Feynman diagrams where changing a particle into an antiparticle just reverses the arrow of time.

Of the four solutions in (5.7) $\Psi^{(1)}_{\text{rest}}$ and $\Psi^{(2)}_{\text{rest}}$ represent particles and $\Psi^{(3)}_{\text{rest}}$ and $\Psi^{(4)}_{\text{rest}}$ represent antiparticles. One remaining question is what the difference between the two particle solutions is and what the difference between the two antiparticle solutions is. Recalling that the spin S_z along the z-axis is

$$S_z = \frac{\hbar}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

shows that

$$S_z\Psi^{(1)}_{\text{rest}} = \frac{\hbar}{2}\Psi^{(1)}_{\text{rest}} \qquad S_z\Psi^{(2)}_{\text{rest}} = -\frac{\hbar}{2}\Psi^{(2)}_{\text{rest}} \qquad S_z\Psi^{(3)}_{\text{rest}} = -\frac{\hbar}{2}\Psi^{(3)}_{\text{rest}} \qquad S_z\Psi^{(4)}_{\text{rest}} = \frac{\hbar}{2}\Psi^{(4)}_{\text{rest}}$$

for particles at rest. However one is not mainly interested in fermions at rest but in moving fermions because one wants to understand scattering events on one side and because there are or may be massless fermions which cannot be brought to rest on the other side.

Starting from the ansatz $\Psi(X) = Ae^{-ik_\mu X^\mu}u(k^\mu)$ for which $\partial_\mu\Psi = -ik_\mu\Psi$ the Dirac equation becomes $(i\hbar\gamma^\mu k_\mu - mc)\Psi = 0$ which is algebraic. Plane waves as a guess for the form of the solution is almost always what one uses in case of a definite direction and magnitude of propagation but completely undefined location. In this ansatz A is a normalization factor, $e^{-ik_\mu X^\mu}$ is the plane wave, and $u(k^\mu)$ is the spin. One can show that $k^\mu = \pm\frac{1}{\hbar}P^\mu$ and will get

$$\begin{aligned}
\Psi^{(1)} &= Ae^{i\frac{P_\mu X^\mu}{\hbar}}\begin{pmatrix}\frac{E}{mc^2}-\frac{P_z}{mc}\\ -\frac{P_x}{mc}-i\frac{P_y}{mc}\\ 1\\ 0\end{pmatrix} & \Psi^{(2)} &= Ae^{i\frac{P_\mu X^\mu}{\hbar}}\begin{pmatrix}-\frac{P_x}{mc}+i\frac{P_y}{mc}\\ \frac{E}{mc^2}+\frac{P_z}{mc}\\ 0\\ 1\end{pmatrix}\\
&= Ae^{i\frac{P_\mu X^\mu}{\hbar}}U^{(1)} & &= Ae^{i\frac{P_\mu X^\mu}{\hbar}}U^{(2)}\\
\Psi^{(3)} &= Ae^{-i\frac{P_\mu X^\mu}{\hbar}}\begin{pmatrix}0\\ 1\\ -\frac{P_x}{mc}+i\frac{P_y}{mc}\\ -\frac{E}{mc^2}+\frac{P_z}{mc}\end{pmatrix} & \Psi^{(4)} &= Ae^{-i\frac{P_\mu X^\mu}{\hbar}}\begin{pmatrix}1\\ 0\\ -\frac{E}{mc^2}-\frac{P_z}{mc}\\ -\frac{P_x}{mc}-i\frac{P_y}{mc}\end{pmatrix}\\
&= Ae^{-i\frac{P_\mu X^\mu}{\hbar}}V^{(1)} & &= Ae^{-i\frac{P_\mu X^\mu}{\hbar}}V^{(2)}
\end{aligned} \tag{5.8}$$

as the four solutions of the Dirac equation (5.6) in any frame where $U^{(1)}$ and $U^{(2)}$ are particle spinors and $V^{(1)}$ and $V^{(2)}$ are antiparticle spinors. Note that $\Psi^{(i)} \to \Psi^{(i)}_{\text{rest}}$ for $P_x = P_y = P_z = 0$ when the particle or antiparticle is at rest.

The spinors depend on the energy and momentum. This is very different from non-relativistic Quantum Mechanics where solving the Schrödinger equation always gives solutions $\psi = \psi_{\text{space}}(\vec{x}, t)\chi$ with a spatial part depending on $\vec{x}$ and t and a constant spinor χ with no dependence of $\vec{x}$ and t if it is a particle with spin. In relativistic Quantum Field Theory the relativistic spinors become much more complicated and the spinorial part is non-trivially dependent on energy and momentum.

One consequence is that the solutions (5.8) are no eigenstates of S_z unlike the solutions (5.7) in the rest frame. When something is at rest, there is no preferred direction is space, but if it is moving, the direction along which it is moving is special. Thus, if one does not select the spin S_z along the z-axis but the spin $S_{\vec{P}}$ along the direction of the momentum, there might be some simplifications. One can chose coordinates such that the z-axis points in the direction of the momentum with $P_x = P_y = 0$ and $P = P_z$. This leads back to the nice structure where either only the first and third or only the second and fourth component of the spinor is non-zero. Thus, the solutions satisfy

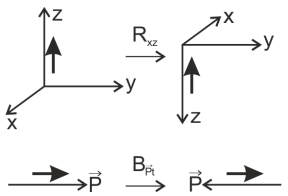
$$S_z\Psi^{(1)} = \frac{\hbar}{2}\Psi^{(1)} \qquad S_z\Psi^{(2)} = -\frac{\hbar}{2}\Psi^{(2)} \qquad S_z\Psi^{(3)} = -\frac{\hbar}{2}\Psi^{(3)} \qquad S_z\Psi^{(4)} = \frac{\hbar}{2}\Psi^{(4)}$$

and are therefore eigenstates of S_z. The general solutions (5.8) are not eigenstates of S_z, because the z-axis is a arbitrary direction.

One usually thinks that if things are rotationally invariant, x, y, z should be interchangeable, but P_x, P_y, P_z appear in very different ways in the solutions (5.8). The reason is that the γ matrices are different for the different coordinates. It is the Dirac equation (5.6) itself which treats x, y, z not the same.

5.5 Helicity and Chirality

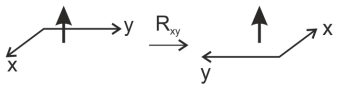
It is often useful to work in terms of eigenstates of spin along the direction of motion which are referred to as helicity states. In Particle Physics *helicity* is defined as the spin operator along the momentum direction.



Characterizing particle states with helicity is almost just like characterizing them by S_z. For instance if a particle has $S_z = +\frac{\hbar}{2}$, one can always rotate the coordinate system so that the same particle gets $S_z = -\frac{\hbar}{2}$. It is still a useful classification if one sticks to one coordinate system. Helicity is similar. If one has a state with positive helicity $S_{\vec{P}} = +\frac{\hbar}{2}$ then one can turn it into a state with negative helicity $S_{\vec{P}} = -\frac{\hbar}{2}$ by boosting in the plane of $\vec{P}$ and t to a frame which is moving faster than the particle such that the momentum of the particle is reversed.

However, if the particle is massless there is no way to reverse $\vec{P}$ with a boost or a change of the coordinate system. So for massless particles their helicity is an unchangeable intrinsic property just like their total spin. In fact for a given massless particle flavor one might as well think of the $S_{\vec{P}} = \pm\frac{\hbar}{2}$ states as two different particles.

The Wigner classification for massive particles to classify the degrees of freedom gives the particle a four-momentum and then asks what transformations are possible without changing the momentum. If the counting of the degrees of freedom is independent of the frame, then one can pick the simplest frame which is the rest frame and can apply the rotations in SO(3) without changing the four-momentum.



For massless particles there is also a four-momentum with the speed of light, but going to the rest frame is not an option. However one can chose a momentum along one of the spatial axes. Selecting the x-axis means that $P^0 = \frac{E}{c}$ and $P^2 = P^3 = 0$, but because $P^\mu P_\mu = m^2c^2 = 0$ for massless particles $P^1 = \frac{E}{c}$ as well. The transformations which do not change the four-momentum are the ones of SO(2). Rotations in the plane perpendicular to the momentum cannot change the helicity.

There are cases where helicity and chirality are describing the same thing, but in general they are not. For purposes of the Standard Model especially when talking about spontaneous symmetry breaking or the weak interaction, one should think in terms of chirality and not in terms of helicity. Chirality has to do with the block structure

$$\sigma^{0i} = \frac{i}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix} \qquad\qquad \sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}$$

of the generators (4.11) for boosts on spinors on the left side and for rotations on spinors on the right side where the indexed 2×2 matrices σ are the Pauli matrices. The upper two components and the lower two components of the four components of a spinor which is a solution of the Dirac equation (5.6) are transformed independently by boosts and by rotations and therefore by all Lorentz transformations because using the exponential map the rough matrix form of the generators is going essentially to descend to the actual transformation matrix. The block diagonal generators leads therefore to block diagonal transformations. Because the first two and the second two components of the spinor Ψ get transformed independently, one can use

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix} = \begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix} \qquad \Psi_+ = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \qquad \Psi_- = \begin{pmatrix}\Psi_3\\ \Psi_4\end{pmatrix} \tag{5.9}$$

where $\Psi_\pm$ are called the Weyl spinors or the *chiral* spinors and determine the *chirality* of the spin-$\frac{1}{2}$ particles. They transform oppositely under boosts and alike under rotations, and they do not mix.

The free Lagrangian for a spinor field Ψ is

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} &= (\hbar c)\bar{\Psi}\gamma^\mu\partial_\mu\Psi + mc^2\bar{\Psi}\Psi \\ &= -\hbar c\left(i\Psi_-^\dagger\partial_\mu\sigma^\mu\Psi_- + i\Psi_+^\dagger\partial_\mu\bar{\sigma}^\mu\Psi_+\right) + mc^2\left(\Psi_-^\dagger\Psi_+ + \Psi_+^\dagger\Psi_-\right)\end{aligned}$$

where $\bar{\Psi} = (\Psi_-^\dagger, \Psi_+^\dagger)$, $\sigma^\mu = (I, \sigma^i)$ and $\bar{\sigma}^\mu = (I, -\sigma^i)$. The mass term $mc^2\bar{\Psi}\Psi$ mixes Ψ_+ and Ψ_-, but this does not happen in the first term. Thus if $m = 0$ then there is no mass term, and the Lagrangian becomes the sum of a term with only Ψ_- and a term with only Ψ_+. This means that it is completely consistent to think of a theory containing only the term Ψ_- or only the term Ψ_+ as long as the particle is massless. However in order to have mass both parts Ψ_- and Ψ_+ are needed. If one of the two parts is zero, the mass term is also zero.

The two Weyl equations $\bar{\sigma}^\mu\partial_\mu\Psi_+ = 0$ and $\sigma^\mu\partial_\mu\Psi_- = 0$ describe each a particle/antiparticle pair for $m = 0$ and have therefore two real degrees of freedom. Working with Ψ_+ and Ψ_- for massless spinors is exactly the same as working with positive or negative helicity states. To see this, one can first align the z-axis of the coordinate system to the direction of the momentum to make $P_x = P_y = 0$. Further follows from $m = 0$ and the mass-shell condition (3.4) $E^2 - P^2c^2 = m^2c^4$ that $E = \pm Pc$. Thus the solutions (5.8) become

$$\Psi^{(1)} = Ae^{i\frac{P_\mu X^\mu}{\hbar}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \Psi^{(2)} = Ae^{i\frac{P_\mu X^\mu}{\hbar}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad \Psi^{(3)} = Ae^{-i\frac{P_\mu X^\mu}{\hbar}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \Psi^{(4)} = Ae^{-i\frac{P_\mu X^\mu}{\hbar}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

and obviously form a basis for (5.9).

However there is a way to define the splitting into Ψ_+ and Ψ_- even before one takes the massless limit. That is what chirality is. Thinking about helicity before taking the massless limit is not useful because one can change helicity by changing the coordinate system. But chirality is not something one can influence by changing the coordinate system because of the block diagonal form of the generators (4.11). Chirality states that if one is in state Ψ_+ one stays in state Ψ_+, and if one is in state Ψ_- one stays in state Ψ_- independent of whatever one does to the coordinates.

Note that everything done here is couched in the particular choice of the γ matrices that has been made. Thus the question is what happens if one picks a different set of γ matrices, and one would like to have a definition of positive and negative chirality independent of the choice of the basis. This is possible using $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ which is written in terms of 2×2 matrices

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{5.10}$$

in the particular representation used here. One can form a projection operator $P_\pm = \frac{1}{2}(I \pm \gamma^5)$ with $P_+\Psi = \Psi_+$ and $P_-\Psi = \Psi_-$. Thus in the example of the chosen γ matrices γ^5 can be used to project an arbitrary four-component spinor onto the chiral pieces Ψ_+ and Ψ_-. The introduction of γ^5 is a way to distinguish positive and negative chirality independent of the choice of the γ matrices. As it will turn out, weak interaction, unlike electromagnetic and strong interaction, will treat Ψ_+ differently than Ψ_-.

6 Interactions

6.1 Local Gauge Invariance

After studying free Lagrangians (5.3) for scalar, vector and spinor fields with their respective equations of motions, the next step is the introduction of interactions. Most interesting is the Dirac Lagrangian because the Standard Model is to a great deal concerned with interactions between matter particles, matter consists of fermions, and fermions are spin-$\frac{1}{2}$ particles. The free or kinetic terms in all three free

Lagrangians typically involve derivatives for velocities as well as mass terms for the rest mass energy which are always quadratic in the field in question. This will turn out to be important.

If two different fields are going to interact, some constant times the product of the two fields describes such an interaction. The product of a field with itself represents a mass term, but the product of two fields means an interaction between the two fields. Examples of terms in the Lagrangian for a scalar field Φ interacting with a spinor field Ψ are $\Phi\Psi$ or $\Phi\,\partial_\mu\Psi$. A priori one might think that one could add two different free Lagrangians and add any interaction terms to the Lagrangian as long as they are Lorentz invariant, and in a certain sense one can, but it turns out that one can describe every experimental observation by only introducing interaction terms that follow from an elegant symmetry principle. These symmetry principles are called *gauge invariance* which reduces the huge number of possible terms to a handful of used terms.

There is a recipe for a gauge theory:

1. Begin from a free Dirac Lagrangian with a global symmetry.
2. Promote the global symmetry to a local gauge symmetry with the addition of a compensating gauge field that itself must transform in a specific way. This will introduce interactions.
3. Allow the gauge field to propagate by adding a field strength term to the Lagrangian.

This recipe is shown step by step in an example for the abelian case. The first step is starting from the free Dirac Lagrangian $\mathcal{L} = \hbar c\bar{\Psi}\gamma^\mu\partial_\mu\Psi + mc^2\bar{\Psi}\Psi$ from (5.3) where $\bar{\Psi} = i\Psi^\dagger\gamma^0$. Further one can transform the spinor field $\Psi \to \Psi' = e^{iq\phi}\Psi$ where the transformation is not a coordinate transformation but just a multiplication with a complex number of magnitude 1 which is constant because q and ϕ are supposed to be constants. It follows $\bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}e^{-iq\phi}$, and it therefore follows that this is a symmetry of the Lagrangian because

$$\mathcal{L} \to \mathcal{L}' = \hbar c\bar{\Psi}e^{-iq\phi}\gamma^\mu\partial_\mu e^{iq\phi}\Psi + mc^2\bar{\Psi}e^{-iq\phi}e^{iq\phi}\Psi = \mathcal{L}$$

since $e^{iq\phi}$ is a constant such that $e^{-iq\phi}\gamma^\mu\partial_\mu e^{iq\phi} = e^{-iq\phi}e^{iq\phi}\gamma^\mu\partial_\mu$. Multiplication with $e^{iq\phi}$ is a global transformation.

In the second step this global transformation is promoted to a local transformation $\Psi \to \Psi' = e^{iq\phi(X^\mu)}\Psi$ where $\phi = \phi(X^\mu)$ can have different values at different points in spacetime. The Lagrangian becomes

$$\mathcal{L} \to \mathcal{L}' = \hbar c\bar{\Psi}e^{-iq\phi}\gamma^\mu\partial_\mu e^{iq\phi}\Psi + mc^2\bar{\Psi}e^{-iq\phi}e^{iq\phi}\Psi$$

but here $e^{-iq\phi}\gamma^\mu\partial_\mu e^{iq\phi} \neq e^{-iq\phi}e^{iq\phi}\gamma^\mu\partial_\mu$ because of the dependence on X^μ. The mass term stays the same because $mc^2\bar{\Psi}e^{-iq\phi}e^{iq\phi}\Psi = mc^2\bar{\Psi}\Psi$, but the whole Lagrangian is not invariant under this transformation. Using the product rule the first term of the Lagrangian becomes

$$\begin{aligned}\hbar c\bar{\Psi}e^{-iq\phi}\gamma^\mu\partial_\mu e^{iq\phi}\Psi &= \hbar c\bar{\Psi}e^{-iq\phi}\gamma^\mu\left[iq\big(\partial_\mu\phi\big)e^{iq\phi}\Psi + e^{iq\phi}\partial_\mu\Psi\right] \\ &= i\hbar cq\bar{\Psi}\gamma^\mu\big(\partial_\mu\phi\big)\Psi + \hbar c\bar{\Psi}\gamma^\mu\partial_\mu\Psi\end{aligned}$$

where only one term causes problems. To get rid of it, the solution is to change the ordinary derivative ∂_μ to the so-called *covariant* derivative $D_\mu = \partial_\mu + iqA_\mu$ where A_μ is a new field such that the Lagrangian becomes

$$\mathcal{L} = \hbar c\bar{\Psi}\gamma^\mu D_\mu\Psi + mc^2\bar{\Psi}\Psi = \hbar c\bar{\Psi}\gamma^\mu\partial_\mu\Psi + iq\hbar c\bar{\Psi}\gamma^\mu A_\mu\Psi + mc^2\bar{\Psi}\Psi$$

which transforms with $\Psi \to \Psi' = e^{iq\phi}\Psi$ using the product rule to

$$\begin{aligned}\mathcal{L}' &= \hbar c\bar{\Psi}e^{-iq\phi}\gamma^\mu\partial_\mu e^{iq\phi}\Psi + iq\hbar c\bar{\Psi}e^{-iq\phi}\gamma^\mu A'_\mu e^{iq\phi}\Psi + mc^2\bar{\Psi}e^{-iq\phi}e^{iq\phi}\Psi \\ &= \hbar c\bar{\Psi}e^{-iq\phi}\gamma^\mu\left[iq\big(\partial_\mu\phi\big)e^{iq\phi}\Psi + e^{iq\phi}\partial_\mu\Psi\right] + iq\hbar c\bar{\Psi}\gamma^\mu A'_\mu\Psi + mc^2\bar{\Psi}\Psi \\ &= \hbar c\bar{\Psi}\gamma^\mu\partial_\mu\Psi + iq\hbar c\bar{\Psi}\gamma^\mu\big(\partial_\mu\phi\big)\Psi + iq\hbar c\bar{\Psi}\gamma^\mu A'_\mu\Psi + mc^2\bar{\Psi}\Psi\end{aligned}$$

where canceling factors $e^{-iq\phi}e^{iq\phi}$ have been removed. If $A'_\mu = A_\mu - \partial_\mu\phi(X^\nu)$ one gets

$$\mathcal{L}' = \hbar c\bar{\Psi}\gamma^\mu\partial_\mu\Psi + iq\hbar c\bar{\Psi}\gamma^\mu A_\mu\Psi + mc^2\bar{\Psi}\Psi = \mathcal{L}$$

which is invariant. The term $iq\hbar c\bar{\Psi}\gamma^\mu A_\mu\Psi$ is an interaction. Starting from a global symmetry turned into a local symmetry did not only lead to a new field A which is called a *gauge field* but also to an

interaction between the original field Ψ and the new field A. The gauge field A however cannot propagate and just constitutes a background. That means if one tries to apply an action principle to get an equation of motion for A, one would not get a very interesting result, but one would certainly want to have an equation of motion for A. If one wants A allow to propagate and to get useful equations of motion for it, one has to give it a kinetic term.

That is what one does in the third step. It turns out that A_μ is a dual vector field and it is clear which kinetic term one has to use. The Lagrangian for a vector field in (5.3) is the Proca Lagrangian $\mathcal{L} = \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\pi}A^\mu A_\mu$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The total Lagrangian becomes

$$\mathcal{L} = \hbar c\bar{\Psi}\gamma^\mu D_\mu\Psi + mc^2\bar{\Psi}\Psi + \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\pi}A^\mu A_\mu$$

but it turns out not to be invariant under the transformation $A_\mu \to A'_\mu = A_\mu - \partial_\mu\phi$. The first two terms are obviously invariant. The third term is $F_{\mu\nu}F^{\mu\nu} = (\partial_\kappa A_\lambda - \partial_\lambda A_\kappa)\eta^{\mu\kappa}\eta^{\nu\lambda}(\partial_\mu A_\nu - \partial_\nu A_\mu)$ without constant factors and represents the kinetic term. It becomes

$$\begin{aligned}
F'_{\mu\nu}F'^{\mu\nu} &= (\partial_\kappa A'_\lambda - \partial_\lambda A'_\kappa)\eta^{\mu\kappa}\eta^{\nu\lambda}(\partial_\mu A'_\nu - \partial_\nu A'_\mu) \\
&= (\partial_\kappa A_\lambda - \partial_\kappa\partial_\lambda\phi - \partial_\lambda A_\kappa + \partial_\lambda\partial_\kappa\phi)\eta^{\mu\kappa}\eta^{\nu\lambda}(\partial_\mu A_\nu - \partial_\mu\partial_\nu\phi - \partial_\nu A_\mu + \partial_\nu\partial_\mu\phi) \\
&= (\partial_\kappa A_\lambda - \partial_\lambda A_\kappa)\eta^{\mu\kappa}\eta^{\nu\lambda}(\partial_\mu A_\nu - \partial_\nu A_\mu) = F_{\mu\nu}F^{\mu\nu}
\end{aligned}$$

which shows that it is invariant because partial derivatives commute such that $\partial_\mu\partial_\nu\phi = \partial_\nu\partial_\mu\phi$. The last term representing the mass of the field A_μ is $A^\mu A_\mu$ without constant factors and becomes

$$A'^\mu A'_\mu = (A_\mu - \partial_\mu\phi)\eta^{\nu\mu}(A_\nu - \partial_\nu\phi)$$

which is not the same as $A^\mu A_\mu$. Thus the last term must vanish, and the mass of the field A_μ must as a consequence be zero. This is an important observation because there are massive gauge fields in the world. (That is where the Higgs mechanism will play a role.) The total Lagrangian is therefore

$$\mathcal{L} = \hbar c\bar{\Psi}\gamma^\mu D_\mu\Psi + mc^2\bar{\Psi}\Psi + \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \tag{6.1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The Lagrangian (6.1) is invariant under local $U = e^{iq\phi}$ transformations with $U^\dagger U = I$. This corresponds to the group U(1) and therefore to electromagnetism. The vector field A_μ is the usual electromagnetic 4-vector potential which transforms under this gauge symmetry as $A_\mu \to A'_\mu = A_\mu - \partial_\mu\phi$. In this context A_μ is called a gauge field which mediates the electromagnetic interaction. One can identify q as the coupling strength of Ψ to the electromagnetic field and therefore q is the electric charge. The tensor $F_{\mu\nu}$ is the electromagnetic field strength.

Electromagnetism comes usually in the form of the E-field and the B-field with the Maxwell equations and so on, but in this form it has no connection to the weak and strong interactions and gravity. However, in the form shown here as a gauge theory with the Lagrangian (6.1) it is the starting point for Quantum Electrodynamics.

The recipe for a gauge theory has been applied to an abelian symmetry. Later it will be extended to non-abelian symmetries, but before that a few clarifications are necessary, because the weak and the strong interactions are based on a non-abelian symmetry. To simplify the non-abelian case, the abelian case can be formulated differently.

When trying to make the symmetry local, the main problem is that the derivative did not satisfy the property needed because $\bar{\Psi}\gamma^\mu\partial_\mu\Psi \to \bar{\Psi}e^{-iq\phi}\gamma^\mu\partial_\mu e^{iq\phi}\Psi \neq \bar{\Psi}e^{-iq\phi}\gamma^\mu e^{iq\phi}\partial_\mu\Psi$. This has been fixed with the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$ which gives $\bar{\Psi}\gamma^\mu D_\mu\Psi \to \bar{\Psi}e^{-iq\phi}\gamma^\mu D'_\mu e^{iq\phi}\Psi = \bar{\Psi}e^{-iq\phi}\gamma^\mu e^{iq\phi}D_\mu\Psi$, and the magic trick is $D_\mu\Psi \to D'_\mu e^{iq\phi}\Psi = e^{iq\phi}D_\mu\Psi$. This is equivalent to what is done in General Relativity where the ordinary derivative ∂_μ does not transform as a tensor under a general coordinate transformation $X^\mu \to X^{\mu'}(X^\mu)$. To fix the problem the covariant derivative $\partial_\mu \to D_\mu + \Gamma...$ has been introduced where Γ represents the Christoffel symbols which play the role of gauge fields in General Relativity. Here the transformation is not a spacetime transformation[5] but in General Relativity it is, and these spacetime transformations are localized. Thus gravitation comes out of a similar argument.

[5]In the Standard Model the transformations are all in an internal space, but there is a way to geometrize all of this. One can take these internal spaces and consider them as part of a hybrid spacetime. This is a program called fiber bundle.

In order to restructure the abelian theory for the non-abelian case, note that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant for the abelian case but is not for the non-abelian case. A different route to getting a gauge invariant field strength is to consider

$$\begin{aligned}
[D_\mu, D_\nu]\Psi &= (\partial_\mu + iqA_\mu)(\partial_\nu + iqA_\nu)\Psi - (\partial_\nu + iqA_\nu)(\partial_\mu + iqA_\mu)\Psi \\
&= (\partial_\mu + iqA_\mu)(\partial_\nu\Psi + iqA_\nu\Psi) - (\partial_\nu + iqA_\nu)(\partial_\mu\Psi + iqA_\mu\Psi) \\
&= \partial_\mu\partial_\nu\Psi + iqA_\mu\partial_\nu\Psi - q^2A_\mu A_\nu\Psi + iq(\partial_\mu A_\nu)\Psi + iqA_\nu\partial_\mu\Psi \\
&- \partial_\nu\partial_\mu\Psi - iqA_\nu\partial_\mu\Psi + q^2A_\nu A_\mu\Psi - iq(\partial_\nu A_\mu)\Psi - iqA_\mu\partial_\nu\Psi \\
&= iq(\partial_\mu A_\nu - \partial_\nu A_\mu)\Psi
\end{aligned}$$

with the consequence $F_{\mu\nu} = -\frac{i}{q}[D_\mu, D_\nu]$ which will be very helpful in the non-abelian case because it is gauge invariant even if it is a non-abelian symmetry. (Note that the Riemann curvature tensor in General Relativity is derived with the commutator of the covariant derivative.)

To summarize the Lagrangian for electromagnetism (6.1) with $F_{\mu\nu} = -\frac{i}{q}[D_\mu, D_\nu]$ is invariant under local U(1) transformations. The interaction term is dictated by the symmetry and it turns out that it agrees with experiments.

6.2 Quantum Chromodynamics

The development for the non-abelian case of the strong interactions will follow a very similar path as the abelian case of electromagnetism. The symmetry is based on SU(3) acting in a three-dimensional space with eight generators instead of U(1) acting in a one-dimensional space with one generator. Because all matter consists of spin-$\frac{1}{2}$ particles, Quantum Chromodynamics will at least be partly described by a spinor field Ψ with four components in spin space but in addition with the three components Ψ_r, Ψ_b, Ψ_g in color space. Color space with the three colors red, blue, green as a basis is three-dimensional and has nothing to do with the spin space which is four-dimensional. One way to keep them separate is that the four components of the spinor are affected but not the three color components if one does a Lorentz transformation, and if one does an SU(3) transformation the three color components are affected but not the four spinor components. Spaces such as the color space are often called iso-spaces or internal vector spaces just to make sure one is not confused between those vector spaces and the vector spaces more related to spacetime. Note however that also in spacetime there are two kinds of vector spaces, one with vector type indices corresponding to tensors and the spin space which both transform under spacetime transformations but transform differently.

Transformations of SU(3) act on the vectors with the three color components. Each of the three components Ψ_r, Ψ_b, Ψ_g is a four-component spinor. The strong interaction only acts on the six flavors of quarks which are the up u and down d, the charm c and strange s, and the top t and bottom b quark. Each of these six types of quarks can come in red r, blue b, and green g which creates a multiplicity of types of quarks. The different flavors of quarks have different mass and electrical charge, but a red up quark and a blue up quark are actually identical except that one is red and one is blue and they just represent different color states. For each flavor of quark there is a flavor of antiquark $\bar{u}$, $\bar{d}$, $\bar{c}$, $\bar{s}$, $\bar{t}$, $\bar{b}$, with the possible colors antired $\bar{r}$, antiblue $\bar{b}$, antigreen $\bar{g}$. Note that red, blue, green as a kind of charge are not three values of the same charge like $+e$ and $-e$ for the electrical charge, but are three distinct types of charge each with two values r, $\bar{r}$ or b, $\bar{b}$ or g, $\bar{g}$. The reason why the strong interactions only act on quarks is that they are the only particles carrying color the same way as the electromagnetic interaction only acts on particles with electric charge. However in nature one never finds single quarks but only bound states of quarks and only bound states of quarks which are colorless.

Quantum Chromodynamics describing the strong interactions is understood as a theory of local SU(3) invariance acting on quarks only. The free Dirac Lagrangian $\mathcal{L} = \hbar c\bar{\Psi}\gamma^\mu\partial_\mu\Psi + mc^2\bar{\Psi}\Psi$ is the starting Lagrangian with $\bar{\Psi} = i\Psi^\dagger\gamma^0$ which includes apart from a transpose in spin space also a transpose in color space because Ψ is also a vector in color space. This Lagrangian is invariant under global

$$\Psi \to \Psi' = e^{-i\frac{q}{\hbar c}\lambda\cdot\phi}\Psi \qquad\qquad A = e^{ig_A v^A}$$

where the equation on the right side is the exponential map (4.1) from Lie algebras to Lie group elements. Comparing the two equations shows that λ corresponds to the vector g_A of matrix generators while ϕ

corresponds to the vector v^A of paramters. Because SU(3) has eight generators, λ and ϕ have both eight elements. The factor $g = \frac{q}{\hbar c}$ is a constant which will turn out to determine the coupling strength for all the three types of color charge[6] because SU(3) mixes the three colors and symmetry requires that all couple equally. The quantity ϕ will get a dependence on spacetime when going from the global to the local symmetry while the quantity λ is a set of fixed generators. Giving λ indices would make λ^a_{ij} where the a is an index in the generator space and the i and j are in the color space. Because SU(3) is the related group of transformations, λ have to be 3×3 matrices and thus needs two indices i and j, but because there are eight of them, the index a is needed. This is a generalization of the case U(1) to the non-abelian group SU(3) with multiple generators.

In the next step, the global symmetry is turned into the local one

$$\Psi \to \Psi' = e^{-i\frac{q}{\hbar c}\lambda\cdot\phi(X^\mu)}\Psi$$

with $\phi = \phi(X^\mu)$. The Lagrangian is no longer invariant under local SU(3), but with the covariant derivative $D_\mu = \partial_\mu + ig\,\lambda \cdot A_\mu$ instead of ∂_μ similar to the case in electromagnetism one can make the Lagrangian invariant under local SU(3). One needs eight gauge fields A_μ, one for each generator. Introducing the gauge fields A_μ does not make the Lagrangian invariant, one first has to figure out how the eight fields A_μ transform to make it invariant. Thus, the covariant derivative has to transform as $D_\mu\Psi \to D'_\mu\Psi' = e^{-ig\lambda\cdot\phi}D_\mu\Psi$ for a spacetime dependent ϕ. Using

$$\begin{aligned} D_\mu\Psi = \partial_\mu\Psi + ig\,\lambda\cdot A_\mu\Psi \to \partial\mu\Psi' + ig\,\lambda\cdot A'_\mu\Psi' &= \partial_\mu\left(e^{-ig\lambda\cdot\phi}\Psi\right) + ig\,\lambda\cdot A'_\mu e^{-ig\lambda\cdot\phi}\Psi \\ &= \partial_\mu\left(e^{-ig\lambda\cdot\phi}\right)\Psi + e^{-ig\lambda\cdot\phi}\partial_\mu\Psi + ig\,\lambda\cdot A'_\mu e^{-ig\lambda\cdot\phi}\Psi \end{aligned}$$

the equation

$$\partial_\mu\left(e^{-ig\lambda\cdot\phi}\right)\Psi + e^{-ig\lambda\cdot\phi}\partial_\mu\Psi + ig\,\lambda\cdot A'_\mu e^{-ig\lambda\cdot\phi}\Psi = e^{-ig\lambda\cdot\phi}\left(\partial_\mu\Psi + ig\,\lambda\cdot A_\mu\Psi\right)$$

is satisfied if $\lambda\cdot A'_\mu = e^{-ig\lambda\cdot\phi}\,\lambda\cdot A_\mu\, e^{ig\lambda\cdot\phi} + \frac{i}{g}\partial_\mu(e^{-ig\lambda\cdot\phi})e^{ig\lambda\cdot\phi}$. Comparing this to the U(1) case where λ was a number shows that the result is much uglier. The fact that λ is a matrix in color space in the non-abelian group for Quantum Chromodynamics does not allow to move factors freely around such that this expression cannot be simplified. The interaction term added to the Lagrangian by $\hbar c\bar\Psi\gamma^\mu D_\mu\Psi$ is $ig\hbar c\bar\Psi\gamma^\mu\lambda\cdot A_\mu\Psi$. It describes the interaction between the quark fields Ψ and the gluons A_μ, thus it describes the strong force.

The last step adds the necessary kinetic terms for the fields A_μ to the Lagrangian in order to let the gauge fields propagate. Starting similar to the abelian case from the massless Proca Lagrangian $\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$ the quantity $F_{\mu\nu}$ becomes

$$\begin{aligned} -\frac{i}{g}[D_\mu, D_\nu]\Psi &= -\frac{i}{g}(\partial_\mu + ig\lambda\cdot A_\mu)(\partial_\nu\Psi + ig\lambda\cdot A_\nu\Psi) + \frac{i}{g}(\partial_\nu + ig\lambda\cdot A_\nu)(\partial_\mu\Psi + ig\lambda\cdot A_\mu\Psi) \\ &= -\frac{i}{g}\partial_\mu\partial_\nu\Psi + \frac{i}{g}\partial_\nu\partial_\mu\Psi + \lambda\cdot A_\nu\partial_\mu\Psi + \lambda\cdot A_\mu\partial_\nu\Psi - \lambda\cdot A_\nu\partial_\mu\Psi - \lambda\cdot A_\mu\partial_\nu\Psi \\ &\quad + \partial_\mu(\lambda\cdot A_\nu)\Psi - \partial_\nu(\lambda\cdot A_\mu)\Psi + ig(\lambda\cdot A_\mu)(\lambda\cdot A_\nu)\Psi - ig(\lambda\cdot A_\nu)(\lambda\cdot A_\mu)\Psi \end{aligned}$$

such that $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu(\lambda\cdot A_\nu) - \partial_\nu(\lambda\cdot A_\mu) + ig[\lambda\cdot A_\mu, \lambda\cdot A_\nu]$. Because it is a non-abelian group, the commutators $[\lambda\cdot A_\mu, \lambda\cdot A_\nu]$ do not vanish. The Lie algebra for SU(3) is $[g_i, g_j] = if^{ijk}g_k$ where the values f^{ijk} are the structure constants of SU(3). The quantities λ are the generators and one can therefore write $[\lambda^a, \lambda^b] = if^{abc}\lambda^c$. The fact that the commutators $[\lambda\cdot A_\mu, \lambda\cdot A_\nu]$ do not vanish has nothing to do with the gauge fields A_μ but comes from the generators λ. Thus with

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu(\lambda\cdot A_\nu) - \partial_\nu(\lambda\cdot A_\mu) + ig[\lambda\cdot A_\mu, \lambda\cdot A_\nu] = \lambda^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - gf^{abc}\lambda^a A^b_\mu A^c_\nu \\ &= \lambda^a\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\right) = \lambda^a F^a_{\mu\nu} \end{aligned}$$

summing over a the kinetic term for the eight fields A^a_μ becomes

$$\begin{aligned} \frac{1}{16\pi}F^a_{\mu\nu}F^{\mu\nu\,a} &= \frac{1}{16\pi}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\right)\left(\partial^\mu A^{\nu\,a} - \partial^\nu A^{\mu\,a} - gf^{ade}A^{\mu\,d}A^{\nu\,e}\right) \\ &= \frac{1}{16\pi}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu\,a} - \partial^\nu A^{\mu\,a}\right) + \text{gluon-gluon interactions} \end{aligned}$$

[6] Also the other three forces (electromagnetism, weak interaction, and gravity) have only one coupling constant each. The coupling constant determines the strength of the force.

where the gluon-gluon interactions which mix the eight gauge fields are

$$-\frac{g}{16\pi}f^{ade}A^d_\mu A^e_\nu(\partial^\mu A^{\nu\,a}-\partial^\nu A^{\mu\,a})-\frac{g}{16\pi}f^{abc}A^{\mu\,b}A^{\nu\,c}(\partial_\mu A^a_\nu-\partial^\nu A^a_\mu)+\frac{g^2}{16\pi}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu\,d}A^{\nu\,e}$$

in addition to the expected kinetic term $\frac{1}{16\pi}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)(\partial^\mu A^{\nu\,a}-\partial^\nu A^{\mu\,a})$ which actually consists of eight terms for $a\in\{1,...,8\}$ but does not mix the gauge fields. Note that the gluon-gluon interactions critically depend on SU(3) being non-abelian such that $f^{abc}\neq 0$. The photons in electromagnetism (6.1) do not interact with each other because U(1) is abelian. The gluon-gluon interactions with three or four gluons bring in a host of new effects including glueballs, confinement, and so on. Unlike the vacuum in electromagnetism the vacuum in Quantum Chromodynamics is vary active because the gauge fields interact heavily. To conclude this application of the recipe for gauge theories and to summarize,

$$\begin{aligned}\mathcal{L}&=\hbar c\bar\Psi\gamma^\mu D_\mu\Psi+mc^2\bar\Psi\Psi+\frac{1}{16\pi}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)(\partial^\mu A^{\nu\,a}-\partial^\nu A^{\mu\,a})\\&-\frac{g}{16\pi}f^{ade}A^d_\mu A^e_\nu(\partial^\mu A^{\nu\,a}-\partial^\nu A^{\mu\,a})-\frac{g}{16\pi}f^{abc}A^{\mu\,b}A^{\nu\,c}(\partial_\mu A^a_\nu-\partial^\nu A^a_\mu)\\&+\frac{g^2}{16\pi}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu\,d}A^{\nu\,e}\end{aligned}\tag{6.2}$$

is the complete Lagrangian for Quantum Chromodynamics.

6.3 Electroweak Gauge Theory

The weak interactions have many peculiar features that sets them apart from the strong and the electromagnetic interactions:

i) Every single matter particle in the Standard Model exhibits weak interactions while only charged particles feel the electromagnetic force and only quarks feel the strong force.
ii) The force mediators are massive unlike the massless photons and gluons.
iii) The weak interactions violate charge conjugation (C), parity (P), the combination of charge conjugation and parity (CP), and time (T).
iv) The weak interactions can change the flavor of the particles and are therefore responsible for decays.

Perhaps the strangest part of the weak interactions is that they are not realized as a symmetry of the Standard Model, at least not at room temperature type energies. It is based on a symmetry that is broken, and this is tied to the fact that the mediators are massive.

In order to use the recipe for gauge theories in line with electromagnetism and Quantum Chromodynamics, one can formulate the weak interactions in terms of a gauge symmetry. This is relevant since at some point in the history of the universe this is how it appeared. A surprising feature of formulating the weak interactions in terms of a gauge symmetry is that this is not possible for the weak force alone but one is forced to unify the weak force with electromagnetism to do so. This is called the electroweak unification. One often hears that the total gauge group of the Standard Model is SU(3) × SU(2) ×U(1) where SU(3) is Quantum Chromodynamics, SU(2) is the weak force, and U(1) is electromagnetism. This is not quite right. The correct groups are $\mathrm{SU(3)}\times\mathrm{SU(2)_L}\times\mathrm{U(1)_Y}$ for high energy where L stands for left chirality and Y for hypercharge, and $\mathrm{SU(3)}\times\mathrm{U(1)_{EM}}$ for low energy where EM stands for electromagnetism. The weak interactions cannot be observed as a manifestly realized symmetry group at low temperature, and $\mathrm{U(1)_Y}$ is not the same as $\mathrm{U(1)_{EM}}$. The group SU(3) is in both cases the symmetry group for Quantum Chromodynamics, and $\mathrm{SU(2)_L}\times\mathrm{U(1)_Y}$ together is the symmetry group for the unified electroweak force.

Thus the starting point is therefore the symmetry group $\mathrm{SU(2)_L}\times\mathrm{U(1)_Y}$ which at some point in the history of the universe has been broken. The free Dirac Lagrangian $\mathcal{L}=\hbar c\bar\Psi\gamma^\mu\partial_\mu\Psi+mc^2\bar\Psi\Psi$ is again the Lagrangian used but Ψ is assumed to be in the form (5.9) as Weyl spinors $\Psi_\pm$. The Dirac Lagrangian becomes $-\hbar c(i\Psi_-^\dagger\partial_\mu\sigma^\mu\Psi_-+i\Psi_+^\dagger\partial_\mu\bar\sigma^\mu\Psi_+)+mc^2(\Psi_-^\dagger\Psi_++\Psi_+^\dagger\Psi_-)$ where $\sigma^\mu=(I,+\sigma^i)$ and $\bar\sigma^\mu=(I,-\sigma^i)$. Note that the mass term mixes Ψ_+ and Ψ_-, but the other two terms do not.

In order to use consistently spinors with four components, the spinors Ψ_R and Ψ_L defined as

$$\Psi_R=\begin{pmatrix}\Psi_+\\0\end{pmatrix}\qquad\qquad\Psi_L=\begin{pmatrix}0\\\Psi_-\end{pmatrix}\qquad\qquad\Psi=\begin{pmatrix}\Psi_+\\\Psi_-\end{pmatrix}=\Psi_R+\Psi_L$$

are used instead of $\Psi_\pm$. As shown above when introducing the matrix γ^5 (5.10), the two projection operators $P_\pm = \frac{1}{2}(I \pm \gamma^5)$ can be used to project an arbitrary four-component spinor Ψ onto the chiral parts $\Psi_R = P_+\Psi$ and $\Psi_L = P_-\Psi$. The matrix γ^5 has the properties

$$\gamma^{5\,2} = I \qquad\qquad \gamma^{5\,\dagger} = \gamma^5 \qquad\qquad \{\gamma^5, \gamma^\mu\} = 0$$

used below. The Dirac Lagrangian becomes $\mathcal{L} = \hbar c(\bar\Psi_L + \bar\Psi_R)\gamma^\mu\partial_\mu(\Psi_L + \Psi_R) + mc^2(\bar\Psi_L + \bar\Psi_R)(\Psi_L + \Psi_R)$. Applying $\bar\Psi = i\Psi^\dagger\gamma^0$ on the spinor Ψ_R gives

$$\bar\Psi_R = \overline{\frac{1}{2}(I+\gamma^5)\Psi} = i(\frac{1}{2}(I+\gamma^5)\Psi)^\dagger\gamma^0 = i\Psi^\dagger(\frac{1}{2}(I+\gamma^5)^\dagger\gamma^0 = i\Psi^\dagger(\frac{1}{2}(I+\gamma^5)\gamma^0 = i\Psi^\dagger\gamma^0(\frac{1}{2}(I-\gamma^5) = \bar\Psi P_-$$

such that $\bar\Psi_R = \bar\Psi P_-$. The cross terms in the Lagrangian which connect a left and a right projection leading to expressions like $\bar\Psi_R\gamma^\mu\Psi_L = \bar\Psi P_-\gamma^\mu P_-\Psi = \bar\Psi\gamma^\mu P_+P_-\Psi$ all disappear because after P_- the components to project for P_+ are zero or, more formally, because $\frac{1}{2}(I+\gamma^5)\frac{1}{2}(I-\gamma^5) = \frac{1}{4}(I-(\gamma^5)^2) = 0$. Similarly, terms like $\bar\Psi_L\Psi_L$ in the mass term also disappear because $\bar\Psi_L\Psi_L = \bar\Psi P_+P_-\Psi = 0$. Thus, in the kinetic term of the Lagrangian only terms remain which are combinations of the same chirality, while in the mass term only terms survive with different chirality. The Lagrangian becomes

$$\mathcal{L} = \hbar c\bar\Psi_L\gamma^\mu\partial_\mu\Psi_L + \hbar c\bar\Psi_R\gamma^\mu\partial_\mu\Psi_R + mc^2(\bar\Psi_L\Psi_R + \bar\Psi_R\Psi_L)$$

and looks similar to the Lagrangian with Weyl spinors but is here written in spinors with four components. One can therefore not build a mass term with only left or only right contributions. (This will play a role for the mass of neutrinos because one observes only left-handed neutrinos in nature, but right-handed neutrinos would be needed for the neutrino to form a mass term.)

This Lagrangian is used in the following as the free Lagrangian for the electroweak interaction, and the symmetry group is $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$. In the case of Quantum Chromodynamic with the symmetry group SU(3), triplets with the three components Ψ_r, Ψ_b, Ψ_g were needed to act on. Here doublets are needed for $\mathrm{SU(2)_L}$ to act on and singlets for $\mathrm{U(1)_Y}$ to act on. The doublets are the left-handed flavor doublets

$$\chi_L = \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L, \begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}_L, \begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix}_L, \begin{pmatrix}u\\ d\end{pmatrix}_L, \begin{pmatrix}c\\ s\end{pmatrix}_L, \begin{pmatrix}t\\ b\end{pmatrix}_L$$

containing two particles as components. A SU(2) matrix may therefore change a neutrino ν_e into an electron e. The singlets are the right-handed singlets e_R, μ_R, τ_R, u_R, d_R, c_R, s_R, t_R, b_R. At the time the electroweak unification was worked on, there were no right-handed neutrinos in nature, neutrinos had zero mass, and experimental observations supported that. All the other particles have mass, and therefore there must be right-handed versions if there are left-handed versions. Each component of the doublets and each singlet is described by a four-component spinor, but because they are either left-handed or right-handed they will only have either the upper two or the lower two non-zero components. In order to simplify the theory, only the doublet with the electron e_L and the singlet e_R will be explored because the other doublets and singlets only differ by their mass. The quarks feel the strong interactions and the leptons do not, but this is not relevant when talking about the electroweak interactions. This simplifies the Langrangian to $\mathcal{L} = \hbar c\,\bar\chi_L\gamma^\mu\partial_\mu\chi_L + \hbar c\,\bar e_R\gamma^\mu\partial_\mu e_R$ plus the mass term which can be ignored at this point because it does not generate interaction terms. This Lagrangian is invariant under global $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$ where $\mathrm{SU(2)_L}$ represents isospin and $\mathrm{U(1)_Y}$ hypercharge and where the actual transformations are

$$\begin{aligned}
\mathrm{SU(2)_L}: &\qquad \chi_L \to \chi_L' = e^{-ig\,\vec\sigma\cdot\vec\theta}\chi_L \\
\mathrm{U(1)_Y}: &\qquad \chi_L \to \chi_L' = e^{-ig'Y_{\chi_L}\phi}\chi_L \qquad\qquad e_R \to e_R' = e^{-ig'Y_{e_R}\phi}e_R
\end{aligned}$$

with $g = \frac{q}{\hbar c}$ and with the factors $Y_{\chi L}$ and Y_{eR} to allow χ_L and e_R to carry different amounts of the same hypercharge governed by g'. The transformation for SU(2) which is isomorphic to SO(3) near the identity, uses the Pauli matrices $\vec\sigma$ as generators (4.4) because rotations in three dimensions can be written as complex 2×2 matrices expressed in the Pauli matrices. Here the symmetry group is SU(2) and is not describing rotations but this weird transformation in flavor space. However, the group is the same SU(2), and one should be able to write down the matrix elements of SU(2) using the same structure as used for rotations in three dimensions. The transformations for U(1) is basically the same as for electromagnetism, and is therefore just a phase.

Each of these global symmetries can be promoted to a local symmetry of the Lagrangian by the same method as used for Quantum Chromodynamics and electromagnetism by introducing

$$\partial_\mu \chi_L \to D_\mu \chi_L = \partial_\mu \chi_L + ig\,\vec{\sigma}\cdot\vec{W}_\mu \chi_L + ig' Y_{\chi_L} B_\mu \chi_L$$
$$\partial_\mu e_R \to D_\mu e_R = \partial_\mu e_R + ig' Y_{e_R} B_\mu e_R$$

as covariant derivatives where $\vec{W}_\mu$ are three gauge fields for $\mathrm{SU(2)_L}$ and B_μ is one gauge field for $\mathrm{U(1)_Y}$. The quantities $\vec{\theta}$ and ϕ depend on X^μ for the localized symmetries. The gauge fields transform as

$$\vec{\sigma}\cdot\vec{W}_\mu \to \vec{\sigma}\cdot\vec{W}'_\mu = e^{-ig\,\vec{\sigma}\cdot\vec{\theta}}\,\vec{\sigma}\cdot\vec{W}_\mu\, e^{ig\,\vec{\sigma}\cdot\vec{\theta}} + \frac{i}{g}\partial_\mu(e^{-ig\,\vec{\sigma}\cdot\vec{\theta}})e^{ig\,\vec{\sigma}\cdot\vec{\theta}} \qquad B_\mu \to B'_\mu = B_\mu + \partial_\mu\phi$$

to make the covariant derivatives work as expected. As in Quantum Chromodynamics the SU(2) gauge fields transform in a rather complicated way due to the fact that SU(2) is a non-abelian group.

A kinetic term is introduced for each gauge field using $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$ leading to

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad\qquad F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\epsilon^{abc} W^b_\mu W^c_\nu$$

where ϵ^{abc} are the SU(3) and equally SU(2) Lie algebra structure constants (4.3). The terms $\vec{W}_\mu\chi_L$ and $B_\mu\chi_L$ in the definition of the covariant derivatives show that the gauge fields and the matter fields interact, but the term $W^b_\mu W^c_\nu$ in the kinetic terms show also that the three gauge fields W^a_μ interact with each other in the same way as the gluon fields do in Quantum Chromodynamics.

The three gauge bosons W^1_μ, W^2_μ, W^3_μ acting on the left-handed doublets have an interesting structure. As usual in various areas of Quantum Mechanics one can think of the three gauge bosons in terms of the basis $W^z_\mu = W^3_\mu$ and $W^\pm_\mu = \frac{1}{2}(W^1_\mu \pm iW^2_\mu)$. This gives

$$W^z_\mu \sim \frac{1}{2}\sigma^3 = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix} \qquad W^+_\mu \sim \frac{1}{2}(\sigma^1 + i\sigma^2) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad W^-_\mu \sim \frac{1}{2}(\sigma^1 - i\sigma^2) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

as matrices in flavor space. These matrices acting on doublets result in

$$W^z_\mu\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \qquad W^z_\mu\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}$$
$$W^+_\mu\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad W^+_\mu\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$
$$W^-_\mu\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \qquad W^-_\mu\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

on neutrinos on the left side and on electrons on the right side. Thus, W^+_μ turns a neutrino ν_e into an electron e and therefore has to add electric charge, and W^-_μ turns an electron e into a neutrino ν_e and has to remove electric charge. The signs $\pm$ given to W^+_μ and W^-_μ are correlated with electric charge. These gauge bosons mediate interactions which change the electric charge of the matter involved. The W^z_μ on the other hand does not change the particle flavor, hence does not change the electric charge and is electrically neutral. In other words, there is a neutral boson $W^0 = W^z$ leaving electrical charge unchanged and two electrically charged bosons $W^\pm$ changing electrical charge.

There are two big problems with the electroweak unification:

1) The weak gauge bosons are massive, but $W^a_\mu W^{\mu a}$ in the Proca mass term is not gauge invariant.
2) The mass term for spinors requires both the left and the right parts of Ψ to combine, but the gauge theory for the electroweak interaction transforms the left and the right parts differently.

In other words, neither the electroweak gauge bosons nor the fermions constituting matter can have mass. The electroweak symmetry with SU(2) was dropped for ten years because there was no way to reconcile it with massive gauge bosons until a group of scientists including Peter Higgs showed how it can be saved.

Both problems will be solved with the Higgs mechanism for mass generation. A crucial part of this process is the breaking of $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y} \to \mathrm{U(1)_{EM}}$. Early in the universe everything was massless and $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$ was a symmetry of the Standard Model. As the universe cooled this symmetry group

underwent a process called spontaneous symmetry breaking down to $\mathrm{U}(1)_{\mathrm{EM}}$. In this process three of the bosons gained mass together with the matter particles. It is not the case that $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ with the four generators W_μ^z, $W_\mu^\pm$, B_μ became $\mathrm{U}(1)_{\mathrm{EM}}$ with one symmetry generator B_μ as one might expect, but this is not the case. What happened is that the B_μ from $\mathrm{U}(1)_{\mathrm{Y}}$ mixes with the neutral W_μ^z from $\mathrm{SU}(2)_{\mathrm{L}}$

$$A_\mu = B_\mu \cos\theta_W + W_\mu^z \sin\theta_W \qquad\qquad Z_\mu^0 = -B_\mu \sin\theta_W + W_\mu^z \cos\theta_W$$

where θ_W is the Weinberg mixing angle which can be measured. The linear combination A_μ has no mass and is the photon of electromagnetism, while the linear combination Z_μ^0 has mass and is the neutral boson of the broken weak interaction together with the two electrically charged $W_\mu^\pm$ gauge bosons.

In the unified gauge group $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ it would not make much sense to call this a unified group if the two factors $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{U}(1)_{\mathrm{Y}}$ have completely independent coupling constants g and g'. They are related by $g \sin\theta_W = g' \cos\theta_W$ but in the broken version of the theory it is more useful to know how the couplings to $W^\pm$, Z^0 and γ are related. It turns out that

$$g_\gamma = g \sin\theta_W = g' \cos\theta_W \qquad g_{W^\pm} = g \qquad g_{Z^0} = \frac{g_\gamma}{\sin\theta_W \cos\theta_W} = \frac{g}{\cos\theta_W} = \frac{g'}{\sin\theta_W} \tag{6.3}$$

are these relations. The relation $g_{W^\pm} = g$ is not surprising because these two gauge fields come from the unbroken symmetry group basically unchanged and just gained mass through the process of symmetry breaking while γ and Z^0 got mixed up with gauge fields of both factors of the unbroken symmetry group. Finally, note that given θ_W and either g or g', all the other numbers in (6.3) can be calculated.

6.4 Field View of the Higgs Mechanism

When localizing any symmetry in the steps to build a gauge theory, the spin-1 gauge fields introduced in the covariant derivative cannot have a non-zero mass term in the Proca Lagrangian. Because the weak interactions transform left-handed fermions differently from right-handed fermions, the mass term $mc^2(\bar\Psi_R\Psi_L + \bar\Psi_L\Psi_R)$ in the Dirac Lagrangian cannot be invariant under $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$. In other words, neither the gauge bosons Z^0 and $W^\pm$ nor all the fermions can have mass. Since these particles have mass, the question is where they get the mass from. It is not allowed to put the mass in, but there is no problem if the masses just pop up.

The Higgs mechanism in the context of the weak interaction is very difficult, and it is therefore shown here in a simplified context based on U(1) instead of the much nastier real case for $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$. The starting point is the Lagrangian Φ is $\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)^*(\partial^\mu\Phi)$ for a complex scalar field $\Phi = \Phi_1 + i\Phi_2$. Instead of a mass term which is a quadratic interaction of the field with itself the Lagrangian becomes $\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)^*(\partial^\mu\Phi) - \frac{1}{2}\mu^2\Phi^*\Phi + \frac{1}{4}\lambda^2(\Phi^*\Phi)^2$ where the first term added to the spin-0 kinetic term is a kind of mass term with the wrong sign and the second term added is a quartic self-interaction. The strength of the two terms is controlled by μ and λ. These two self-interaction terms together are called the potential $U(\Phi, \Phi^*)$ giving $\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)^*(\partial^\mu\Phi) + U(\Phi, \Phi^*)$. The wrong sign mass term interpreted as mass means that negative mass squared or $E^2/c^2 - p^2 < 0$. A particle with $c^2 < v^2$ is called a tachyon, but since Φ is a field there is a more sensible way to interpret this fact in field theory.

The Lagrangian with the two self-interaction terms has a global U(1) symmetry such that $\Phi \to e^{i\theta}\Phi$ and $\Phi^* \to e^{-i\theta}\Phi^*$. By localizing this symmetry to $\Phi' = e^{i\theta(X)}\Phi$ and the covariant derivative $D_\mu = \partial_\mu + i\frac{q}{\hbar c}A_\mu$ the gauge field A_μ transforms as $A_\mu \to A'_\mu = A_\mu - \frac{q}{\hbar c}\partial_\mu\theta$. In the last step a kinetic term is added to the Lagrangian for the gauge field such that it becomes

$$\mathcal{L}(\Phi, \Phi^*, A_\mu) = \frac{1}{2}\left[\left(\partial_\mu - \frac{iq}{\hbar c}A_\mu\right)\Phi^*\right]\left[\left(\partial^\mu + \frac{iq}{\hbar c}A^\mu\right)\Phi\right] - \frac{1}{2}\mu^2\Phi^*\Phi + \frac{1}{4}\lambda^2(\Phi^*\Phi)^2 + \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} \tag{6.4}$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. This gives a gauge theory with a massless gauge field.

When working in Particle Physics one is interested in the tiny fluctuations of the fields which are the observable particles. The theory developed here has the three fields Φ, Φ^* and A_μ where $\Phi = \Phi_1 + i\Phi_2$, and one can either take Φ_1 and Φ_2 or Φ and Φ^* as the independent fields. All three fields can be written as $A_\mu = A_{\mu 0} + \delta A_\mu$, $\Phi = \Phi_0 + \delta\Phi$, $\Phi^* = \Phi_0^* + \delta\Phi^*$ where the terms with the δ represent the tiny fluctuations corresponding to observable particles. One gets the background field configurations $A_{\mu 0}$, Φ_0,

Φ_0^* from solving the equations of motion for the Lagrangian $\mathcal{L}(\Phi, \Phi^*, A_\mu)$ classically and treats the small fluctuations afterwards quantum mechanically.

To solve the equations of motion of the Lagrangian $\mathcal{L}(\Phi, \Phi^*, A_\mu)$ for the simplest possible solutions one can look for constant field configurations where all the terms with derivatives automatically vanish. Looking for constant solutions boils down to setting the derivative of the potential function $U(\Phi, \Phi^*)$ with respect to Φ or Φ^* to zero. This gives

$$\frac{\partial U(\Phi, \Phi^*)}{\partial \Phi^*} = -\frac{1}{2}\mu^2\Phi + \frac{1}{2}\lambda^2|\Phi|^2\Phi = 0$$

with the following solutions:

i) One solution is $A_\mu = 0$ and $\Phi = 0$ leading to $A_\mu = 0 + \delta A_\mu$ and $\Phi = 0 + \delta\Phi$. This is not a very interesting solution because plugging $A_\mu = 0+\delta A_\mu$ and $\Phi = 0+\delta\Phi$ into the Lagrangian $\mathcal{L}(\Phi, \Phi^*, A_\mu)$ gives the same Lagrangian but with $A_\mu \to \delta A_\mu$, $\Phi \to \delta\Phi$, $\Phi^* \to \delta\Phi^*$.

ii) Another less trivial solution is $A_\mu = 0$ and $\Phi = \Phi_0$ where Φ_0 satisfies $|\Phi|^2 = \frac{\mu^2}{\lambda^2}$. The specific choice $\Phi_{10} = \frac{\mu}{\lambda}$ and $\Phi_{20} = 0$ is made here for Φ and leads to $A_\mu = 0 + \delta A_\mu$, $\Phi_1 = \frac{\mu}{\lambda} + \delta\Phi_1$, $\Phi_2 = 0 + \delta\Phi_2$ which gives an interesting solution to be studied further.

Plugging $A_\mu = 0+\delta A_\mu \equiv A_\mu$, $\Phi_1 = \frac{\mu}{\lambda}+\delta\Phi_1 \equiv \frac{\mu}{\lambda}+\eta$, $\Phi_2 = 0+\delta\Phi_2 \equiv \beta$ into the Lagrangian $\mathcal{L}(\Phi, \Phi^*, A_\mu)$ gives

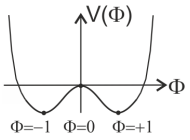
$$\begin{aligned}\mathcal{L} = &\Big[\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \mu^2\eta^2\Big] + \Big[\frac{1}{2}(\partial_\mu\beta)(\partial^\mu\beta)\Big] + \Big[\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\Big(\frac{q}{\hbar c}\frac{\mu}{\lambda}\Big)^2 A_\mu A^\mu\Big] \\ &+ \Big\{\frac{q}{\hbar c}\big[\eta(\partial_\mu\beta) - \beta(\partial_\mu\eta)\big]A^\mu + \frac{\mu}{\lambda}\Big(\frac{q}{\hbar c}\Big)^2\eta(A_\mu A^\mu) + \frac{1}{2}\Big(\frac{q}{\hbar c}\Big)^2\big(\beta^2+\eta^2\big)A_\mu A^\mu \\ &+ \lambda\mu\big(\eta^3 + \eta\beta^2\big) + \frac{1}{4}\lambda^2\big(\eta^4 + 2\eta^2\beta^2 + \beta^4\big)\Big\} + \Big(\frac{\mu}{\lambda}\frac{q}{\hbar c}\Big)(\partial_\mu\beta)A^\mu - \Big(\frac{\mu^2}{2\lambda}\Big)^2\end{aligned} \tag{6.5}$$

where the fields A_μ, η and β depend on position. (Note that this is a simplified version and not the real version for the electroweak force which is even more complicated.) This Lagrangian describes these three fields with the properties:

- The term describing η is $\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \mu^2\eta^2$ in Klein-Gordon form, and $m_\eta = \sqrt{2}\frac{\hbar}{c}\mu$ is therefore the mass of the scalar field η.
- The term describing β is $\frac{1}{2}(\partial_\mu\beta)(\partial^\mu\beta)$ also in Klein-Gordon form but without mass term, and the scalar field β is therefore massless.
- The term $\frac{1}{2}(\frac{q}{\hbar c}\frac{\mu}{\lambda})^2 A_\mu A^\mu$ gives the gauge field A^μ the mass $m_A = 2\sqrt{\pi}\frac{q\mu}{\lambda c^2}$.
- All three fields are interacting with each other in complicated ways.

The mass for the gauge field has not been put in by hand but it came from looking at a particular background of the field Φ. If $\frac{\mu}{\lambda}$ would not have been there, the gauge field would not have mass. The complex field Φ is called the *Higgs field*.

6.5 Symmetry Breaking View of the Higgs Mechanism



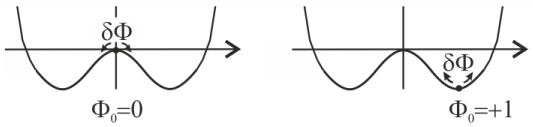
It is possible for a physical theory to have a symmetry even though the experience of this physical theory does not reflect that symmetry. As an example a scalar field Φ with the Lagrangian $\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)$ is used where the potential is $V(\Phi) = -\frac{1}{2}\Phi^2 + \frac{1}{4}\Phi^4$. The field is real such that $\Phi(X^\mu)$ has one degree of freedom, and the extrema are a maximum at $\Phi = 0$ and two minima at $\Phi = \pm 1$ if the field is constant. The potential is an even function in Φ whose graph has therefore the symmetry $V(-\Phi) = V(\Phi)$. It becomes the same when mirrored at the axis $\Phi = 0$. However, through this mirroring process the solution $\Phi = 0$ stays the same while the two solutions $\Phi = \pm 1$ are exchanged. In perturbative physics one does not think about the whole field configuration but more about small fluctuations. One starts with a solution Φ_0 of the classical equation and perturb in small fluctuations about that solution such that one gets $\Phi(X) = \Phi_0 + \delta\Phi(X)$. Putting this into the Lagrangian gives

$$\begin{aligned}\mathcal{L}(\delta\Phi) &= \partial_\mu(\Phi_0 + \delta\Phi)\partial^\mu(\Phi_0 + \delta\Phi) + \tfrac{1}{2}(\Phi_0 + \delta\Phi)^2 - \tfrac{1}{4}(\Phi_0 + \delta\Phi)^4 \\ &= \partial_\mu\delta\Phi\partial^\mu\delta\Phi + \tfrac{1}{2}(\Phi_0 + \delta\Phi)^2 - \tfrac{1}{4}(\Phi_0 + \delta\Phi)^4\end{aligned}$$

where Φ_0 is one of the three static solutions 0 and ± 1. Thus, one can look at different Φ_0 and gets

$$\mathcal{L}_0 = \partial_\mu \delta\Phi \partial^\mu \delta\Phi + \tfrac{1}{2}\delta\Phi^2 - \tfrac{1}{4}\delta\Phi^4$$
$$\mathcal{L}_{+1} = \partial_\mu \delta\Phi \partial^\mu \delta\Phi + \tfrac{1}{2}(1+\delta\Phi)^2 - \tfrac{1}{4}(1+\delta\Phi)^4$$

for $\Phi_0 = 0$ and $\Phi_0 = +1$.



The Lagrangian $\mathcal{L}_0$ is the same as the Lagrangian for Φ and has the same symmetry because $\mathcal{L}_0(\delta\Phi) = \mathcal{L}_0(-\delta\Phi)$ is still valid. The Lagrangian $\mathcal{L}_{+1}$, however, is no longer symmetric because $\mathcal{L}_{+1}(\delta\Phi) \neq \mathcal{L}_{+1}(-\delta\Phi)$. The symmetry is said to be broken. Because $\Phi = 0$ is not changed by $\Phi \to -\Phi$, but $\Phi = \pm 1$ is, the fluctuations about these solutions reflect these different situations. Note that the given potential in both cases is symmetric. The underlying physics is defined by the symmetric potential, but the experience of physics is determined by the small fluctuations. This is the context in which one can say that there is a symmetry but this symmetry has in some sense been broken. It is still there but one does not see it manifestly.

The Higgs field with the Lagrangian (6.4) is a complex field, and $U(\Phi) = -\frac{1}{2}\mu^2\Phi^*\Phi + \frac{1}{4}\lambda^2(\Phi^*\Phi)^2$ is the potential. This is similar to the above example showing how a symmetry can look broken, but the symmetry in the Higgs field is continuous and not discrete as in the example. The Lagrangian for the solution $\Phi = 0$ and $A_\mu = 0$ had turned out to have the same symmetry properties as the original Lagrangian, but the solution $\Phi_1 = \frac{\mu}{\lambda}$, $\Phi_2 = 0$ and $A_\mu = 0$ with the fluctuations $\delta\Phi_1 = \eta$, $\delta\Phi_2 = \beta$ and $\delta A_\mu = A_\mu$ resulted in the Lagrangian (6.5) with the term $\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \mu^2\eta^2$ for the massive scalar field η, the term $\frac{1}{2}(\partial_\mu\beta)(\partial^\mu\beta)$ for the massless scalar field β, the term $\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\frac{q}{\hbar c}\frac{\mu}{\lambda})^2 A_\mu A^\mu$ for the massive vector field A_μ, and with various interaction terms. This also broke a symmetry but a continuous symmetry.

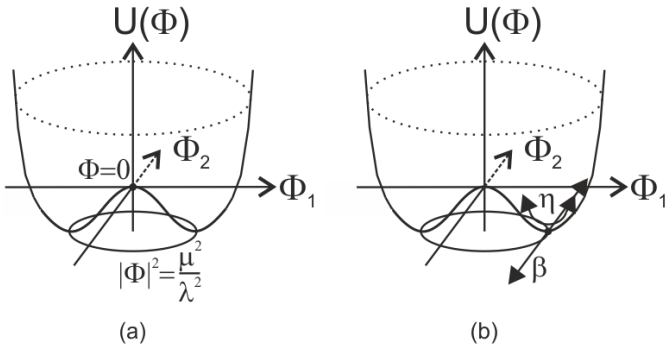


Figure 6.1: The Higgs potential in the form of a Mexican hat

The potential $U(\Phi) = -\frac{1}{2}\mu^2\Phi^*\Phi + \frac{1}{4}\lambda^2(\Phi^*\Phi)^2$ shown in figure 6.1 is commonly called a "Mexican hat" potential, and it looks similar to the broken symmetry example above but the symmetry is now a rotation on a unit circle in the complex plane. There is an obvious maximum at $\Phi = 0$ analogously to the broken symmetry example, but the minima are the circle $|\Phi|^2 = \mu^2/\lambda^2$ building a continuous set of solutions and not just the two discrete points $\Phi = \pm 1$. These solutions are illustrated in (a). The specific solution $\Phi_1 = \frac{\mu}{\lambda}$ and $\Phi_2 = 0$ is presented in (b) with the two fluctuation fields η and β drawn as arrows. The fluctuations η of Φ_1 move up and down the sides of the valley (along the Φ_1-axis), while the fluctuations β of Φ_2 move along the valley on the flat circle of the potential minima. (Note that instead of the point $\Phi_1 = \mu/\lambda$ any other point on the circle with $|\Phi|^2 = \mu^2/\lambda^2$ could have been chosen.)

There is a significance for the fact that η moves up and down while β moves on a flat curve. The mass term is quadratic in the field itself and there is no other field involved. One way to figure out what mass term might result from a potential is to look at the second derivative $d^2U(\Phi)/d\Phi^2|_{\Phi=\Phi_0}$. Given the Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + V(\Phi)$ where the potential is something like $V(\Phi) = e^{ia\Phi}$ it is not obvious that this field has a mass but expanded to the power series $A + B(\Phi-\Phi_0) + C(\Phi-\Phi_0)^2 + D(\Phi-\Phi_0)^3 + ...$ the second derivative is $d^2U(\Phi)/d\Phi^2|_{\Phi=\Phi_0} = 2C$. This is the mass term, and mass is nothing other than the quadratic interaction of the field with itself. In other words, concavity which is what governs

the quadratic term (the second derivative) determines the mass squared. The fluctuations in η have positive mass squared, the fluctuations in β have zero mass squared, and a fluctuation about $\Phi = 0$ has negative mass squared. Negative mass squared just shows an instability, and fluctuating just means rolling away from the starting point. In a field-theory context, massive or massless fluctuations imply stability and tachyonic fluctuations imply instability. Physics without any particles is called a vacuum, and the solutions studied here are different vacua. The solution $\Phi = 0$ is unstable, and as soon as a particle exists it will flow down into the valley. The set of solutions $|\Phi|^2 = \mu^2/\lambda^2$ on the other hand is a circle of stable vacua.

In the above discussion of the Higgs mechanism U(1) has been used instead of the real electroweak symmetry because it makes the calculations as well as the visualization easier. This means that the symmetry is $\Phi \to e^{i\theta}\Phi$. But the fields are now η and β. Thus, the original gauge symmetry becomes a shift of β. It is not a multiplication by a complex number because β is a real scalar field now, but it is a shift $\beta \to \beta + \delta\beta$. A gauge transformation does not change the physical results. In electromagnetism one talks about the electromagnetic fields E and B because they are the physical reality while one can always change the potentials by a gauge transformation. Thus, there is this massless scalar field β that can fluctuate but one is free to change the value of β using $\beta \to \beta+\delta\beta$ back to what it was before without changing the physical content of the theory. That means if β makes a fluctuation $+1$, for example, one can always set it back to zero by such a shift, and one can therefore gauge away any fluctuations of β. These are non-physical fluctuations. Thus, one loses the massless field β by setting it to zero and keeps only the massive scalar field η and the massive gauge field A_μ.

Summarizing, the Mexican hat potential is the Higgs potential, Φ is the Higgs field, η is the massive Higgs boson, and β is a massless Goldstone boson. The Higgs boson has been detected experimentally, and Goldstone bosons always appear if there is a continuous symmetry. The Higgs mechanism gives mass to the gauge fields A_μ of a spontaneously broken gauge symmetry through the coupling to an extra Higgs field Φ which has its own particle η. The two-polarization massless spin-1 field A_μ eats – as it is formulated – the spin-0 Goldstone boson β to get its third polarization degree of freedom which is required when the mass of the gauge field is not zero. One loses one real degree of freedom because β can be gauged away, but one gains one real degree of freedom with the mass for the gauge field A_μ.

The Higgs mechanism has been illustrated with the simplest possible symmetry group U(1) but can be generalized to the breaking of $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y} \to \mathrm{U(1)_{EM}}$ which explains the masses of the $W^\pm$ and Z^0 bosons in the Standard Model. However, not only the gauge bosons are supposed to be massless because of the process for building a gauge theory but also the fundamental matter fermions are taken to be massless for $\mathrm{SU(2)_L}$ invariance. By coupling one can generate effective masses for these as well. One can include a term $\Phi\bar{\Psi}_L\Psi_R$, but this is an interaction of the Higgs field Φ with the fermion field Ψ and not a mass term. The mass term gets created through $\Phi \to \Phi_0 + \delta\Phi$ and $\Phi_0\bar{\Psi}_L\Psi_R + \delta\Phi\bar{\Psi}_L\Psi_R$. The term $\Phi_0\bar{\Psi}_L\Psi_R$ is a mass term because Φ_0 is a constant. Thus the Higgs mechanism solves both mass problems.

6.6 Evolution of the Higgs Potential

The question remains how the Higgs field got into the unstable solution. The factors μ and λ in the Higgs potential are coupling constants similar to the electric charge which is the coupling constant for electromagnetism. It turned out that coupling constants are not constant. The charge of an electron is not a constant. For all practical purposes it is constant because one actually never does experiments in electromagnetism that get close enough to a charged particle to see a difference. If one probes closer and closer to an elementary particle which takes higher and higher energies one would discover that the values of charges change. All couplings vary with energy scales.

Thus, changing the factors μ and λ in $U(\Phi) = -\frac{1}{2}\mu^2\Phi^*\Phi + \frac{1}{4}\lambda^2(\Phi^*\Phi)^2$ changes the shape of the potential. Maybe they change sign. One can imagine that early in the universe when the universe was very hot and everything was highly energetic there was only one solution $\Phi = 0$ and it was stable. But as the universe cooled everything dropped to lower energy densities such that the Higgs potential eventually could have taken the form it has today where the solution $\Phi = 0$ became unstable and is no longer the only solution. The Higgs field could no longer stay at $\Phi = 0$ and rolled down. It had to go somewhere on the circle of the new continuous set of solutions, but the direction was completely arbitrary. Wherever it went is where it is today, but it was completely undetermined. It was spontaneous what direction it picked to

go. Early in the universe there was a stable vacuum with $\Phi = 0$ and what happened later was a so-called vacuum decay into one of the vacua at $|\Phi|^2 = \mu^2/\lambda^2$.

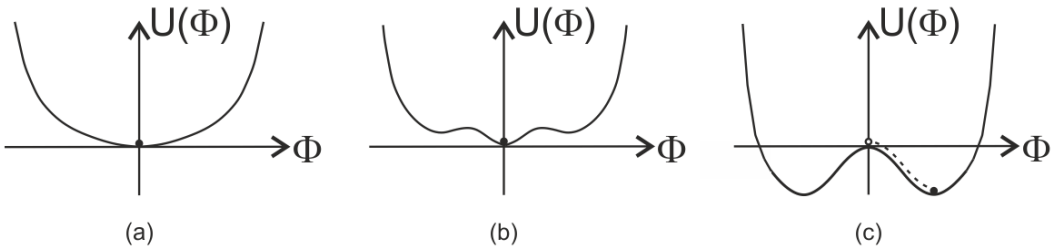
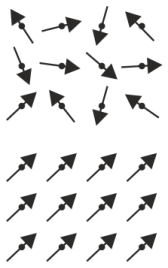


Figure 6.2: Evolution of the Higgs potential in the history of the universe

The way that the Higgs potential could have evolved over the history of the universe is depicted in figure 6.2. In an early stage (a) the lowest energy solution was symmetric $\rightarrow \mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$. In a later stage (b) it was still symmetric $\rightarrow \mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$. In the last stage (c) however the lowest energy solutions have broken symmetry $\rightarrow \mathrm{U(1)_{EM}}$.



Spontaneous symmetry breaking is a known phenomenon in physics. Given a lattice of spins (magnetic dipols, for example) where spins tend to align or anti-align, these spins wiggle randomly at high temperatures. When lowering the temperature more and more the shaking of the spins becomes eventually so slow that they start to align. Two neighbors align and start building a domain and the direction is completely random. Picking the direction was completely spontaneous. The unaligned state of the system was more symmetric and enjoys a full SO(3) invariance. If it is completely random, one can rotate it arbitrarily and it looks the same. As soon as the system is cooled down enough to create alignment, there is only a SO(2) invariance. One can rotate the system about the plane perpendicular to the direction of the alignment. This is spontaneous symmetry breaking. If one takes a field approximation of this system, then one can describe the physics of this system with a Higgs mechanism because the Higgs mechanism is not only useful in Particle Physics but also in condensed matter physics describing phase transitions.

Because the universe is so large that if the Higgs field takes a value in one place another region may be causally disconnected because communication is limited by the speed of light. Thus, if the whole Higgs mechanism takes place over a short enough time frame then there is no reason why the alignment in one place should be the same as in another place, and one might end up just like in magnetic materials with domain walls. In the case of magnetism one understands the domain walls. In the context of breaking a gauge symmetry, it has been discovered that these domain walls give rise to magnetic monopoles because magnetic monopoles are nothing but topologically non-trivial configurations of the gauge field.